Robust Decision Models with Estimation and Prediction

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Stochastic Programming Society (SPS)
Virtual Seminar Series

Robust Decision Models with Estimation and Prediction

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Agenda

- Challenges of Optimizing Decisions Under Uncertainty
- A Tour of Robust Decision Models
- Estimating Parameters of Disparity Metric
- Asymmetric Uncertainty Sets
- Predictive Model with Side Information
- Numerical Studies

Challenges of Optimizing Decisions Under Uncertainty

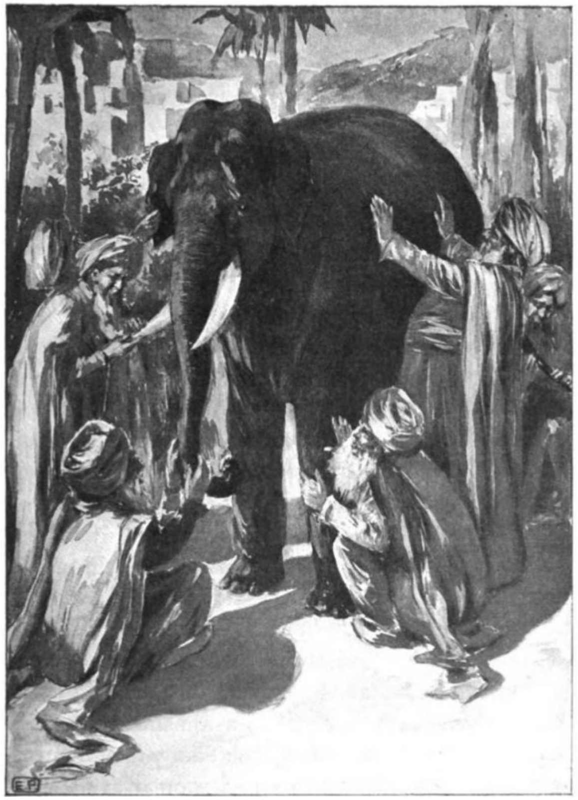
"All Models are Wrong"

– George Box, JASA 1976

Expertise on Decisions Making under Uncertainty

- **Applied Probabilist**
  - Model decision problem as a stochastic process.
  - Focus on formulating stochastic process with analytical tractability.
- **Decision Theorist**
  - Rank various decisions under uncertainty.
  - Focus on rational agents with extensive computational powers.
- **Statistician**
  - Estimate parameters of stochastic process from data.
  - Focus on convergence analysis as data increases.
- **Optimization Expert**
  - Obtain the best solution for the decision problem
  - Focus on efficiency of algorithm and optimality of the solution.

The Blind Men and the Elephant



"The moral of the parable is that humans have a tendency to claim absolute truth based on their limited, subjective experience as they ignore other people's limited, subjective experiences which may be equally true." - Wikipedia

- Illustrator unknown - From *The Heath readers by grades*, D.C. Heath and Company (Boston).

A Portfolio Management Problem

- Investment in N stocks.
- Uncertain future stock returns $\tilde{\boldsymbol{r}}$.
- Portfolio allocation $\boldsymbol{x} \in \mathbb{R}^N$.
- Future portfolio returns: $\tilde{\boldsymbol{r}}^\top \boldsymbol{x}$.

A Portfolio Management Problem

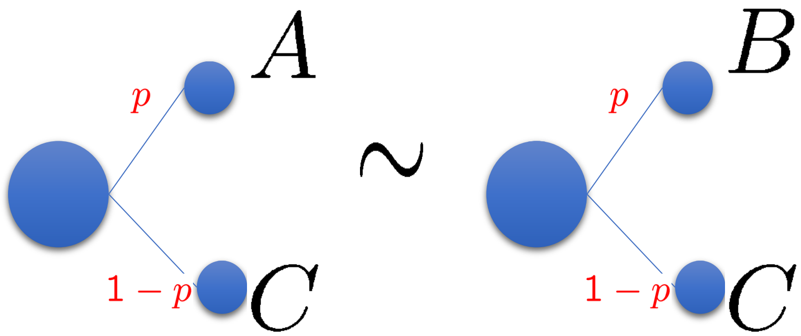
- **Applied Probablist**
  - Model stock returns as a multivariate normal distributed random variable.

$$\tilde{\boldsymbol{r}} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$
$$\tilde{\boldsymbol{r}}^\top \boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}^\top \boldsymbol{x}, \boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x})$$

A Portfolio Management Problem

- **Decision Theorist**
  - Rational agents follows Expected Utility Theory (EUT)
  - Consistent with the following Axioms (von Neumann/Morgenstern)
    - **Complete Ordering**
    - **Transitivity**
    - **Independence**
    - **Continuity**

$$A \sim B$$



A Portfolio Management Problem

- **Decision Theorist**
  - Rank portfolio decisions based on an agent with Constant Absolute Risk Aversion- Exponential Utility.

$$u_\kappa^{-1}\left(\mathbb{E}\left[u_\kappa\left(\tilde{\boldsymbol{r}}^\top \boldsymbol{x}\right)\right]\right) = \boldsymbol{\mu}^\top \boldsymbol{x} - \frac{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}}{2\kappa}$$

where

$$u_\kappa(r) = -\exp(-r/\kappa)$$

A Portfolio Management Problem

- **Statistician**

Given empirical distribution:

$$\tilde{\boldsymbol{r}} \sim \widehat{\mathbb{P}}$$

Estimate mean and covariance by maximizing likelihood:

$$\hat{\boldsymbol{\mu}} = \mathbb{E}_{\widehat{\mathbb{P}}}\left[\tilde{\boldsymbol{r}}\right], \hat{\boldsymbol{\Sigma}} = \mathbb{E}_{\widehat{\mathbb{P}}}\left[(\tilde{\boldsymbol{r}} - \hat{\boldsymbol{\mu}})(\tilde{\boldsymbol{r}} - \hat{\boldsymbol{\mu}})^\top\right]$$

A Portfolio Management Problem

- **Optimization Expert**
  - Obtain optimal portfolio decision by solving a tractable convex quadratic optimization problem.

$$\begin{aligned} \max \quad & \hat{\boldsymbol{\mu}}^\top \boldsymbol{x} - \frac{\boldsymbol{x}^\top \widehat{\boldsymbol{\Sigma}} \boldsymbol{x}}{2\kappa} \\ \text{s.t.} \quad & \mathbf{1}^\top \boldsymbol{x} = 1 \end{aligned}$$

A Portfolio Management Problem

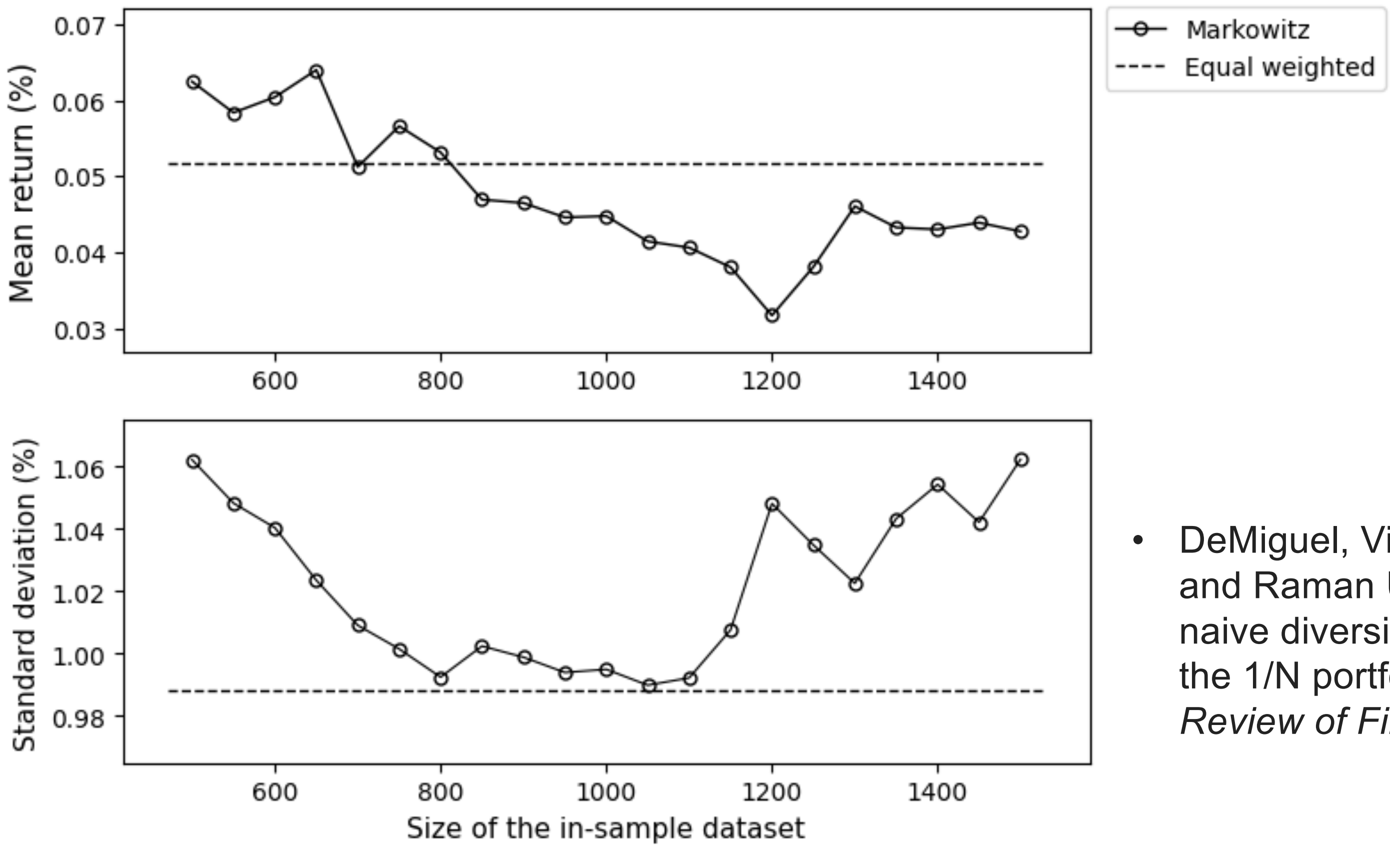
- **Optimization Expert**
  - Constraint on expected return:

$$\begin{aligned} \text{min} \quad & \boldsymbol{x}^\top \widehat{\boldsymbol{\Sigma}} \boldsymbol{x} \\ \text{s.t.} \quad & \widehat{\boldsymbol{\mu}}^\top \boldsymbol{x} = \tau \\ & \mathbf{1}^\top \boldsymbol{x} = 1 \end{aligned}$$

Expected return of equal weighted portfolio:

$$\tau = \frac{1}{N} \widehat{\boldsymbol{\mu}}^\top \mathbf{1}$$

Optimizer’s Curse: Inferior Performance



- DeMiguel, Victor, Lorenzo Garlappi, and Raman Uppal. "Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy?." *The Review of Financial Studies* (2009).

Agenda

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- Estimating Parameters of Disparity Metric
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- Predictive Model with Side Information
- Numerical Studies

A Deterministic Decision Model

$$V = \begin{cases} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & f(\boldsymbol{x}, \hat{\boldsymbol{z}}) \leq \tau \\ & \boldsymbol{x} \in \mathcal{X} \end{cases}$$

- f: attribute function.
- $\boldsymbol{x} \in \mathcal{X}$: here-and-now decisions.
- $\hat{\boldsymbol{z}} \in \mathcal{Z}$: Specified input vector, where $\mathcal{Z}$ is the support set.

On Attribute Function

- Often expressed as an optimization problem over wait-and-see or recourse variables.

$$f(\boldsymbol{x}, \boldsymbol{z}) = \begin{cases} \text{min} & \boldsymbol{d}^\top \boldsymbol{y} \\ \text{s.t.} & \mathbf{B}\boldsymbol{y} \geq \mathbf{A}(\boldsymbol{z})\boldsymbol{x} + \boldsymbol{b}(\boldsymbol{z}) \\ & \boldsymbol{y} \in \mathbb{R}^{N_1} \end{cases}$$

- A decision model can have multiple attribute functions

$$V = \begin{cases} \text{min} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & f_m(\boldsymbol{x}, \hat{\boldsymbol{z}}) \leq \tau_m \quad \forall m \in [M] \\ & \boldsymbol{x} \in \mathcal{X} \end{cases}$$

Budgeted Robust Optimization Model

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & f(\boldsymbol{x}, \boldsymbol{z}) \leq \tau \quad \forall \boldsymbol{z} \in \mathcal{Z}(\gamma) \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

- $\mathcal{Z}(\gamma)$: uncertainty set: subset of support set that contains the specified input vector

$$\mathcal{Z}(\gamma) = \{\boldsymbol{z} \in \mathcal{Z} \mid \Delta(\boldsymbol{z}, \hat{\boldsymbol{z}}) \leq \gamma\}.$$

- $\Delta(\mathrm{z}, \hat{\mathrm{z}})$:Disparity metric
- γ: Disparity budget parameter

On Disparity Metric

- Evaluate distance from specified input.
- Typically based in L-p norm:

$$\Delta(z, \hat{z}) = \|\mathbf{D}^{-1}(z - \hat{z})\|_p.$$

- $\mathbf{D}$: Deviation Matrix: Determine the shape of the uncertainty set
- The choice of L-p norm has the impact on its tractability.
  - p=1 is associated with the uncertainty set of “The Price of Robustness”.

Budgeted Robust Optimization Model

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & f(\boldsymbol{x}, \boldsymbol{z}) \leq \tau \quad \forall \boldsymbol{z} \in \mathcal{Z}(\gamma) \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

- Exact or safe tractable approximations available.
  - Many works on tractable reformulations
  - Facilitated by algebraic modeling language such as RSOME.

RSOME – Robust Stochastic Optimization Made Easy

RSOME

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- User Guide
- Examples
- About

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The current version of RSOME supports deterministic, robust optimization and distributionally robust optimization problems. In the default configuration, RSOME relies on open-source solvers imported from the `scipy.optimize` package to solve linear programming (`linprog()`) and mixed-integer linear programming (`milp`) problems. Besides the default solver, RSOME also provides interfaces for other open-source and commercial solvers. Detailed information of these solver interfaces is presented in the following table.

Solver	License type	Required version	RSOME interface	Second-order cone constraints	Exponential cone constraints	Semidefiniteness constraints
scipy.optimize	Open-source	>= 1.9.0	`lpg_solver`	No	No	No
CyLP	Open-source	>= 0.9.0	`clp_solver`	No	No	No
OR-Tools	Open-source	>= 7.5.7466	`ort_solver`	No	No	No
ECOS	Open-source	>= 2.0.10	`eco_solver`	Yes	Yes	No
Gurobi	Commercial	>= 9.1.0	`grb_solver`	Yes	No	No
Mosek	Commercial	>= 10.0.44	`msk_solver`	Yes	Yes	Yes
CPLEX	Commercial	>= 12.9.0.0	`cpx_solver`	Yes	No	No
COPT	Commercial	>= 7.2.2	`cpt_solver`	Yes	Yes	Yes

Budgeted Robust Optimization Model

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & f(\boldsymbol{x}, \boldsymbol{z}) \leq \tau \quad \forall \boldsymbol{z} \in \mathcal{Z}(\gamma) \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

- Question:
  - How do we specify the disparity budget parameter?
    - Statistical Guarantee?
    - Interpretability?

Relation to Chance Constraint

Suppose $\tilde{z} \sim \mathbb{P}$, then

$$f(x, z) \leq \tau \quad \forall z \in \mathcal{Z}(\gamma)$$
$$\Downarrow$$
$$\mathbb{P}[f(x, \tilde{z}) > \tau] \leq \mathbb{P}[\Delta(\tilde{z}, \hat{z}) > \gamma]$$

- Robust constraint as a tractable approximation of chance constraints
- Caveat:
  - Bound is weak and depends on attribute function
  - Probability distribution is often not known.

Satisficing Decision Model

- Simon (1955, 1959)
  - Human behavior to achieve at least some minimum level of a particular variable, but which does not necessarily maximize its value. Coined the term **satisfice**. (Satisfy + Suffice).
- Lanzilloti (1958)
  - Managers are primarily concerned about target returns on investment.
- Mao (1970)
  - Manager's perception of risk: the prospect of not meeting some target rate of return.

Satisficing Decision Model

P-Model:

$$\begin{aligned} \max \quad & \mathbb{P}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) \leq \tau\right] \\ \text{s.t.} \quad & \boldsymbol{c}^\top \boldsymbol{x} \leq V_1 \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

- Charnes and Cooper. Operations Research. 1963. Deterministic Equivalents for Optimizing and Satisficing under Chance Constraints.
- Computationally intractable

Tractable Satisficing Models

- D. Brown and M. Sim. 2009. Satisficing Measures for Analysis of Risky Positions. *Management Science*.
- D. Brown, E. De Giorgi and M. Sim. 2012. Aspirational Preferences and their Representation by Risk Measures. *Management Science*.
- D.Z. Long, M. Sim and M. Zhou 2022. Robust Satisficing. *Operations Research*.
- M Sim, Q Tang, M Zhou, T Zhu. 2024. The Analytics of Robust Satisficing - Predict, Optimize, Satisfice, then Fortify. *Operations Research*.

Budgeted Robust Satisficing Model

$$\begin{aligned} \max \quad & \gamma \\ \text{s.t.} \quad & f(\boldsymbol{x}, \boldsymbol{z}) \le \tau \qquad \forall \boldsymbol{z} \in \mathcal{Z}(\gamma) \\ & \boldsymbol{c}^\top \boldsymbol{x} \le V_1 \\ & \boldsymbol{x} \in \mathcal{X}, \ \ \gamma \ge 0 \end{aligned}$$

- $V_1 \ge V$: the specified target.
  - Target is less abstract than budget parameter
- Maximize the budget parameter, subject to budgeted robust feasibility
  - Quasiconcave maximation problem
  - Binary search on budget parameter

Tolerated Robust Optimization Model

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & f(\boldsymbol{x}, \boldsymbol{z}) \le \tau + \kappa \Delta(\boldsymbol{z}, \hat{\boldsymbol{z}}) \quad \forall \boldsymbol{z} \in \mathcal{Z} \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

- $\kappa \in [0, +\infty]$: disparity tolerance parameter.
- Violation of attribute constraint can be tolerated
- Magnitude of violation is proportional to the distance from the specified input vector

Relation to Profiled Chance Constraint

Suppose $\tilde{\boldsymbol{z}} \sim \mathbb{P}$, then

$$f(\boldsymbol{x}, \boldsymbol{z}) \leq \tau + \kappa \Delta(\boldsymbol{z}, \hat{\boldsymbol{z}}) \quad \forall \boldsymbol{z} \in \mathcal{Z}$$

$$\Downarrow$$

$$\mathbb{P}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) > \tau + \kappa r\right] \leq \mathbb{P}\left[\Delta(\tilde{\boldsymbol{z}}, \hat{\boldsymbol{z}}) > r\right] \quad \forall r > 0$$

- Ensure that greater violation of constraint might occur at lower probability.
- May not provide probabilistic guarantee of constraint feasibility.

Tolerated Robust Satisficing Model

$$
\begin{aligned}
\min \quad & \kappa \\
\text{s.t.} \quad & f(\boldsymbol{x}, \boldsymbol{z}) \le \tau + \kappa \Delta(\boldsymbol{z}, \hat{\boldsymbol{z}}) \quad \forall \boldsymbol{z} \in \mathcal{Z} \\
& \boldsymbol{c}^\top \boldsymbol{x} \le V_1 \\
& \boldsymbol{x} \in \mathcal{X}, \ \ \kappa \ge 0
\end{aligned}
$$

- $V_1 \ge V$: the specified target.
  - Target is less abstract than tolerance parameter
- Minimize the tolerance parameter, subject to tolerated robust feasibility
  - Limit violation of constraint as much as possible
  - Convex optimization problem

Unified Robust Optimization Framework

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & f(\boldsymbol{x}, \boldsymbol{z}) \leq \tau + \kappa(\Delta(\boldsymbol{z}, \hat{\boldsymbol{z}}) - \gamma)^+, \quad \forall \boldsymbol{z} \in \mathcal{Z} \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

- Dual parameter specifications
  - $\gamma \in [0, +\infty]$: disparity budget.
  - $\kappa \in [0, +\infty]$: disparity tolerance.
- Recovers budgeted robust optimization model if $\kappa = \infty$.
- Recovers tolerated robust optimization model if $\gamma = 0$.

Probabilistic Guarantee

Suppose $\tilde{z} \sim \mathbb{P}$, then

$$f(x, z) \leq \tau + \kappa(\Delta(z, \hat{z}) - \gamma)^{+} \quad \forall z \in \mathcal{Z}$$

$$\Downarrow$$

$$\mathbb{P}\left[f(x, \tilde{z}) > \tau + \kappa r\right] \leq \mathbb{P}\left[\Delta(\tilde{z}, \hat{z}) > \gamma + r\right] \quad \forall r > 0$$

- Ensure that greater violation of constraint might occur at lower probability.
- Also provide probabilistic guarantee of constraint feasibility.

Unified Robust Satisficing Framework

- Specify two targets: $V_2 \geq V_1 \geq V$
- Step 1: Maximize budget as much as possible:

$$\gamma^\star = \begin{cases} \max & \gamma \\ \text{s.t.} & f(\boldsymbol{x}, \boldsymbol{z}) \leq \tau \qquad \forall \boldsymbol{z} \in \mathcal{Z}(\gamma) \\ & \boldsymbol{c}^\top \boldsymbol{x} \leq V_1 \\ & \boldsymbol{x} \in \mathcal{X}, \ \gamma \geq 0 \end{cases}$$

- Step 2: Minimize tolerance as much as possible:

$$\begin{array}{ll} \min & \kappa \\ \text{s.t.} & f(\boldsymbol{x}, \boldsymbol{z}) \leq \tau + \kappa(\Delta(\boldsymbol{z}, \hat{\boldsymbol{z}}) - \gamma^\star)^+, \ \ \forall \boldsymbol{z} \in \mathcal{Z} \\ & \boldsymbol{c}^\top \boldsymbol{x} \leq V_2 \\ & \boldsymbol{x} \in \mathcal{X}, \kappa \geq 0 \end{array}$$

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Estimating Parameters of Disparity Metric

- Robust models are characterized by the disparity metric:

$$\begin{aligned} \text{min} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & f(\boldsymbol{x}, \boldsymbol{z}) \le \tau + \kappa(\Delta(\boldsymbol{z}, \hat{\boldsymbol{z}}) - \gamma)^+, \quad \forall \boldsymbol{z} \in \mathcal{Z} \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

- Suppose we restrict disparity to L-p norm:

$$\Delta(z, \hat{z}) = \|\mathbf{D}^{-1}(z - \hat{z})\|_p.$$

- Can we estimate the parameters of the disparity metric from data?
  - What would be the objective function of such estimation procedure?

Probabilistic Guarantee

Suppose $\boldsymbol{x}$ is feasible in:

$$f(\boldsymbol{x}, \boldsymbol{z}) \leq \tau + \kappa(\|\mathbf{D}^{-1}(\boldsymbol{z} - \hat{\boldsymbol{z}})\|_p - \gamma)^+, \quad \forall \boldsymbol{z} \in \mathcal{Z}$$

and

$$\mathbb{E}_{\mathbb{P}}\left[\psi(\|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}})\|_p)\right] \leq 1.$$

ψ: nonnegative, convex and increasing deviation penalty function.

By Markov inequality,

$$\begin{aligned} \mathbb{P}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) > \tau + \kappa r\right] &\leq \mathbb{P}\left[\|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}})\|_p > r)\right] \\ &\leq \mathbb{P}\left[\psi(\|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}})\|_p) > \psi(r))\right] \\ &\leq 1/\psi(r) \quad \forall r \geq 0. \end{aligned}$$

Probabilistic Guarantee

By specifying the deviation penalty function ψ and ensuring that

$$\mathbb{E}_{\mathbb{P}}\left[\psi(\|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}})\|_p)\right] \leq 1,$$

we have:

$$\mathbb{P}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) > \tau + \kappa r\right] \leq 1/\psi(r) \quad \forall r \geq 0.$$

Suppose $\psi(r) = r^7$, then

$$\mathbb{P}\left[f(\boldsymbol{x}, \tilde{\boldsymbol{z}}) > \tau + \kappa r\right] \leq r^{-7} \quad \forall r \geq 0.$$

Volume of Confidence Set

Define confidence set:

$$\mathcal{B}(\hat{\boldsymbol{z}}, \mathbf{D}, \gamma) = \left\{ \boldsymbol{z} \in \mathbb{R}^L \mid \|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}})\|_p \leq \gamma \right\}.$$

Volume of confidence set:

$$\mathsf{Vol}(\mathcal{B}(\hat{\boldsymbol{z}}, \mathbf{D}, \gamma)) = \det(\mathbf{D})\gamma^L \mathsf{Vol}(\mathcal{B}(\mathbf{0}, \mathbf{I}, 1)).$$

- The volume of the norm ball is related to the size of the uncertainty set.
- The larger the size of the uncertainty set, the more conservative the robust solution might become.

Minimum Volume Confidence Set (MVCS)

$$\begin{aligned} \inf \quad & \det(\mathbf{D}) \\ \text{s.t.} \quad & \mathbb{E}_{\mathbb{P}}\left[\psi(\|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}})\|)\right] \leq 1 \\ & \hat{\boldsymbol{z}} \in \mathbb{R}^L,\ \mathbf{D} \in \mathbb{S}^L_{++}. \end{aligned}$$

Ensures probabilistic guarantee.

Volume of confidence set.

- Key idea in estimating the parameters of the deviation metric:
  - Minimizing the volume of the confidence set, while providing the probabilistic guarantee.
  - Estimating based on providing the least conservative solution with the desired probabilistic guarantee.

MVCS Estimation Problem

Consider the data-driven setting. We have a set of historical data

$$\mathcal{D} = \{(\omega, z_\omega) \mid \omega \in [\Omega]\},$$

where z_ω is the observed outcome of the input vector in scenario ω.

MVCS Estimation Problem

$$\begin{aligned} \inf \quad & \det(\mathbf{D}) \\ \text{s.t.} \quad & \mathbb{E}_{\widehat{\mathbb{P}}}\left[\psi(\|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}})\|)\right] \leq 1 \\ & \mathbf{D} \succeq \text{Diag}(\boldsymbol{r}) \\ & \hat{\boldsymbol{z}} \in \mathbb{R}^L, \; \mathbf{D} \in \mathbb{S}^L_{++}. \end{aligned}$$

$$\widehat{\mathbb{P}}[\tilde{\boldsymbol{z}} = \boldsymbol{z}_\omega] = \frac{1}{\Omega} \quad \forall \omega \in \Omega$$

- Regularization vector $\boldsymbol{r}$ to avoid degeneration.
- Without it, $\mathbf{D}$ may degenerate to a singular matrix, when there is enough data, *i.e.* Ω is small,

MVCS Estimation Problem

$$
\begin{aligned}
\inf \quad & \det(\mathbf{D}) \\
\text{s.t.} \quad & \mathbb{E}_{\hat{\mathbb{P}}}\left[\psi(\|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}})\|)\right] \leq 1 \\
& \mathbf{D} \succeq \text{Diag}(\boldsymbol{r}) \\
& \hat{\boldsymbol{z}} \in \mathbb{R}^L, \ \mathbf{D} \in \mathbb{S}^L_{++}.
\end{aligned}
$$

Not a Convex Optimization Problem

$\Updownarrow$

$$
\begin{aligned}
\max \quad & \sqrt[L]{\det(\mathbf{Q})} \\
\text{s.t.} \quad & \mathbb{E}_{\hat{\mathbb{P}}}\left[\psi(\|\mathbf{Q}\tilde{\boldsymbol{z}} - \boldsymbol{q}\|)\right] \leq 1 \\
& \mathbf{Q} \preceq \text{Diag}^{-1}(\boldsymbol{r}) \\
& \boldsymbol{q} \in \mathbb{R}^L, \ \mathbf{Q} \in \mathbb{S}^L_{+}.
\end{aligned}
$$

Convert to Convex Optimization via change of variables:

$$\mathbf{D} \rightarrow \mathbf{Q}^{-1}, \quad \hat{\boldsymbol{z}} \rightarrow \mathbf{D}\boldsymbol{q}$$

- Max root-determinant optimization problem with semidefinite constraints
- Root-determinant is concave and SOCP representable - Supported in RSOME
- Deviation penalty function is conic representable – Power of exponential cone

Deviation Metric Norm and Penalty Function

Deviation metric based on L-p_1 norm:

$$\|\cdot\| = \|\cdot\|_{p_1}.$$

Deviation penalty based on p_2 power function:

$$\psi(r) = r^{p_2}.$$

Estimation problem:

$$\begin{array}{ll} \inf & \det(\mathbf{D}) \\ \text{s.t.} & \mathbb{E}_{\widehat{\mathbb{P}}}\left[\left(\left\|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}})\right\|_{p_1}\right)^{p_2}\right] \leq 1 \\ & \hat{\boldsymbol{z}} \in \mathbb{R}^L,\ \mathbf{D} \in \mathbb{S}^L_{++}. \end{array}$$

RSOME Code

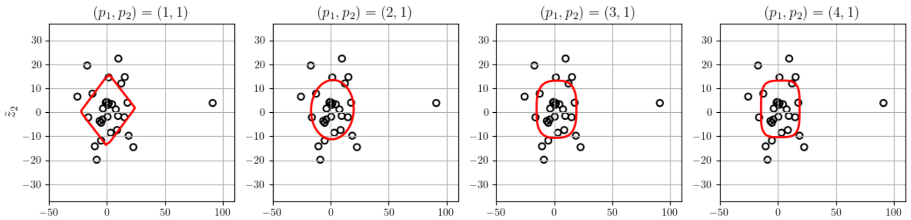
```python
import numpy as np
import rsome as rso
from rsome import ro
from rsome import cpt_solver as cpt
```

```python
N, L = z.shape

model = ro.Model()
q = model.dvar(L)
u = model.dvar(N)
v = model.dvar(N)
Q = model.dvar((L, L))

model.max(rso.rootdet(Q))
model.st((1/N) * u.sum() <= 1)
model.st((1/L) * rso.power(v, p2) <= u)
l_norm = lambda x: rso.norm(x, p1)
for n in range(N):
    model.st(l_norm(Q@z[n] - q) <= v[n])

model.solve(cpt, display=display)

return q.get(), Q.get()
```

$$
\begin{aligned}
\max \quad & \sqrt[L]{\det(\mathbf{Q})} \\
\text{s.t.} \quad & \mathbb{E}_{\widehat{\mathbb{P}}}\left[(\|\mathbf{Q}\tilde{\boldsymbol{z}} - \boldsymbol{q}\|_{p_1})^{p_2}\right] \leq 1 \\
& \boldsymbol{q} \in \mathbb{R}^L, \ \mathbf{Q} \in \mathbb{S}^L_+.
\end{aligned}
$$

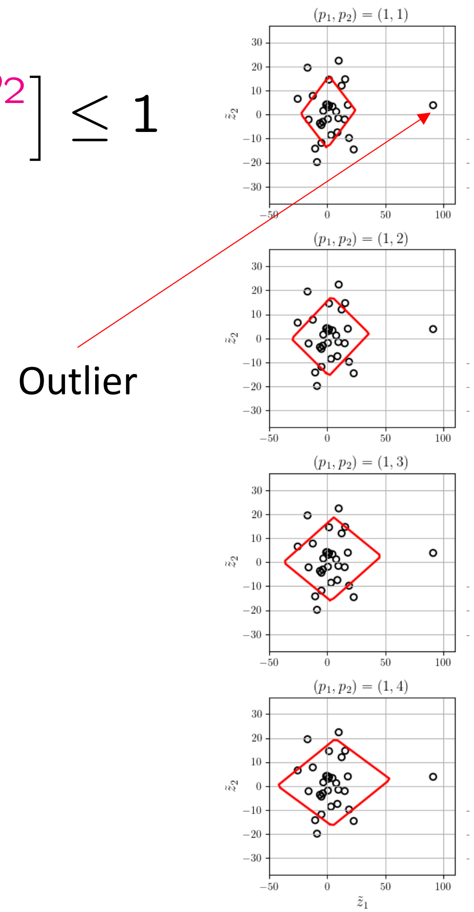
Shape of Uncertainty Set via p_1

$$
\begin{aligned}
\text{inf} \quad & \det(\mathbf{D}) \\
\text{s.t.} \quad & \mathbb{E}_{\widehat{\mathbb{P}}}\left[\left\|\mathbf{D}^{-1}(\tilde{z} - \hat{z})\right\|_{p_1}\right] \le 1 \\
& \hat{z} \in \mathbb{R}^L,\ \mathbf{D} \in \mathbb{S}^L_{++}.
\end{aligned}
$$



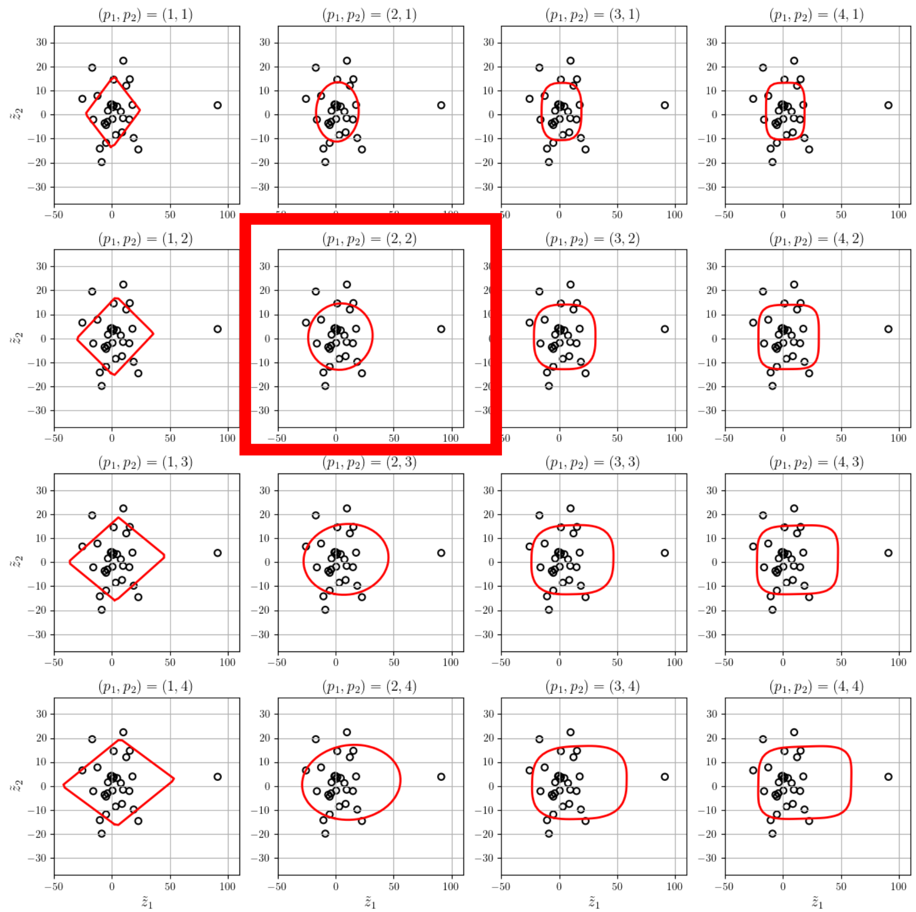
Outlier Sensitivity to Estimation via p_2

$$\begin{aligned} \inf \quad & \det(\mathbf{D}) \\ \text{s.t.} \quad & \mathbb{E}_{\widehat{\mathbb{P}}}\left[\left(\left\|\mathbf{D}^{-1}(\tilde{z} - \hat{z})\right\|_1\right)^{p_2}\right] \leq 1 \\ & \hat{z} \in \mathbb{R}^L,\ \mathbf{D} \in \mathbb{S}^L_{++}. \end{aligned}$$



Uncertainty Set and Outlier Sensitivity

$$
\begin{aligned}
\inf \quad & \det(\mathbf{D}) \\
\text{s.t.} \quad & \mathbb{E}_{\widehat{\mathbb{P}}}\left[\left(\left\|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \widehat{\boldsymbol{z}})\right\|_{p_1}\right)^{p_2}\right] \leq 1 \\
& \widehat{\boldsymbol{z}} \in \mathbb{R}^L, \ \mathbf{D} \in \mathbb{S}^L_{++}.
\end{aligned}
$$



Case of $p_1 = p_2 = 2$

$$\widehat{\boldsymbol{z}}^\star = \mathbb{E}_{\widehat{\mathbb{P}}}[\tilde{\boldsymbol{z}}]$$

$$\mathbf{D}^\star = \mathbb{E}_{\widehat{\mathbb{P}}}\left[(\tilde{\boldsymbol{z}} - \widehat{\boldsymbol{z}}^\star)(\tilde{\boldsymbol{z}} - \widehat{\boldsymbol{z}}^\star)^\top\right]^{1/2}$$

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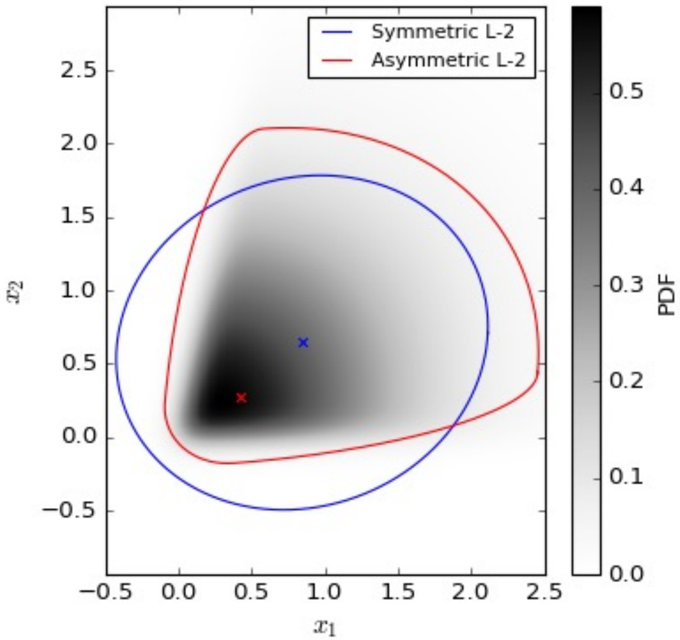
Asymmetric Confidence Set

$$\mathcal{B}(\hat{\boldsymbol{z}}, \mathrm{D}, \underline{\boldsymbol{\sigma}}, \bar{\boldsymbol{\sigma}}, \gamma) = \left\{ \boldsymbol{z} \in \mathbb{R}^L \mid \|(\mathrm{Diag}^{-1}(\underline{\boldsymbol{\sigma}})\mathrm{D}^{-1}(\boldsymbol{z} - \hat{\boldsymbol{z}}))^- + (\mathrm{Diag}^{-1}(\bar{\boldsymbol{\sigma}})\mathrm{D}^{-1}(\boldsymbol{z} - \hat{\boldsymbol{z}}))^+\| \leq \gamma \right\}.$$



How shall we estimate $\hat{\boldsymbol{z}}, \mathbf{D}, \underline{\boldsymbol{\sigma}}, \bar{\boldsymbol{\sigma}}$?

MCVS for Asymmetric Confidence Set

$$\mathrm{Vol}(\mathcal{B}(\hat{\boldsymbol{z}}, \mathbf{D}, \underline{\boldsymbol{\sigma}}, \bar{\boldsymbol{\sigma}}, \gamma)) \propto \prod_{\ell \in [L]} \frac{\underline{\sigma}_\ell + \bar{\sigma}_\ell}{2} \cdot \det(\mathbf{D})$$

$$\begin{aligned} \inf \quad & \prod_{\ell \in [L]} \frac{\underline{\sigma}_\ell + \bar{\sigma}_\ell}{2} \cdot \det(\mathbf{D}) \\ \text{s.t.} \quad & \mathbb{E}_{\widehat{\mathbb{P}}}\left[\psi(\|(\mathrm{Diag}^{-1}(\underline{\boldsymbol{\sigma}})\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}}))^- + (\mathrm{Diag}^{-1}(\bar{\boldsymbol{\sigma}})\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}}))^+\|)\right] \leq 1 \\ & \mathbf{D} \succeq \mathrm{Diag}(\boldsymbol{r}) \\ & \hat{\boldsymbol{z}} \in \mathbb{R}^L, \ \underline{\boldsymbol{\sigma}}, \bar{\boldsymbol{\sigma}} \in \mathbb{R}^L_{++}, \ \mathbf{D} \in \mathbb{S}^L_{++}. \end{aligned}$$

Not representable as a convex optimization problem!

MCVS for Asymmetric Confidence Set

$$
\begin{aligned}
\inf \quad & \prod_{\ell \in [L]} \frac{\underline{\sigma}_\ell + \bar{\sigma}_\ell}{2} \cdot \det(\mathbf{D}) \\
\text{s.t.} \quad & \mathbb{E}_{\hat{\mathbb{P}}}\left[\psi(\|(\mathrm{Diag}^{-1}(\underline{\boldsymbol{\sigma}})\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}}))^{-} + (\mathrm{Diag}^{-1}(\bar{\boldsymbol{\sigma}})\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}}))^{+}\|)\right] \leq 1 \\
& \mathbf{D} \succeq \mathrm{Diag}(\boldsymbol{r}) \\
& \hat{\boldsymbol{z}} \in \mathbb{R}^L, \ \underline{\boldsymbol{\sigma}}, \bar{\boldsymbol{\sigma}} \in \mathbb{R}^L_{++}, \ \mathbf{D} \in \mathbb{S}^L_{++}.
\end{aligned}
$$

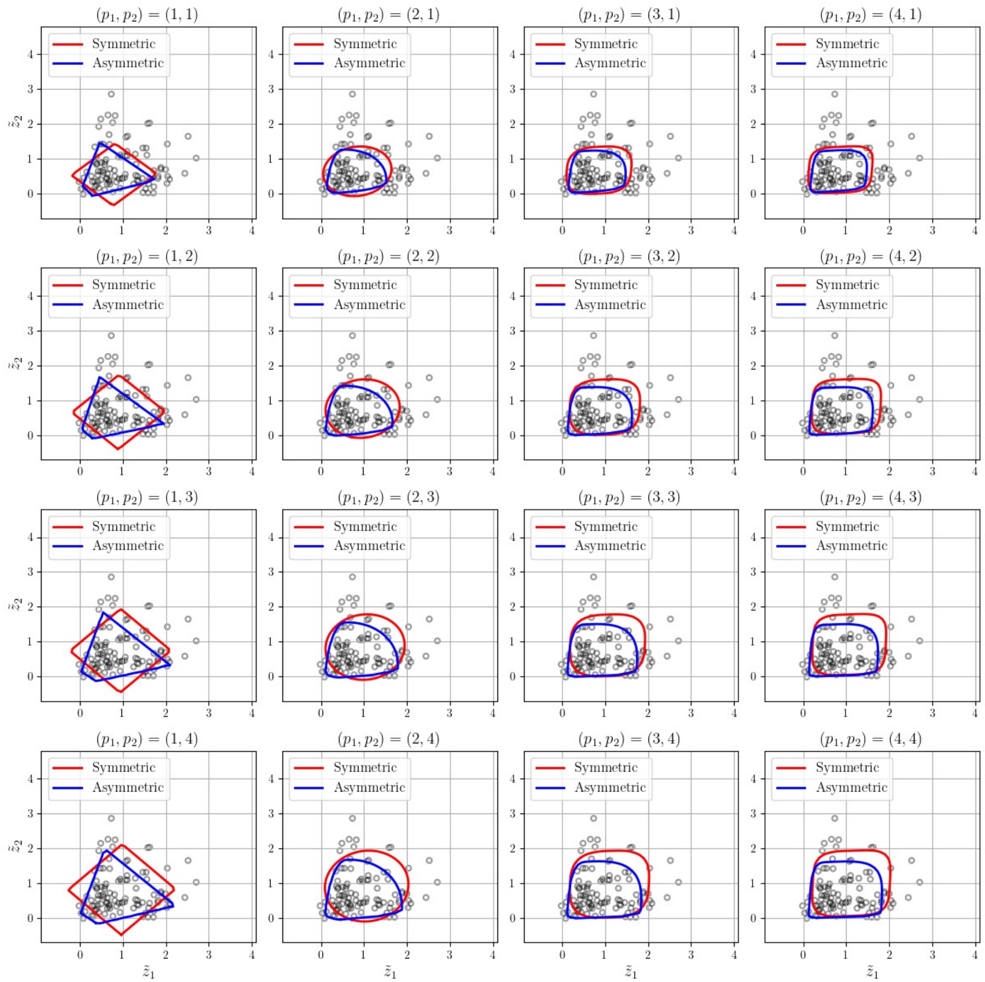
An alternating optimization strategy:

$$
\begin{aligned}
\inf \quad & \prod_{\ell \in [L]} \frac{\underline{\sigma}_\ell + \bar{\sigma}_\ell}{2} \cdot \det(\mathbf{D}) \\
\text{s.t.} \quad & \mathbb{E}_{\hat{\mathbb{P}}}\left[\psi(\|(\mathrm{Diag}^{-1}(\underline{\boldsymbol{\sigma}})\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}}))^{-} + (\mathrm{Diag}^{-1}(\bar{\boldsymbol{\sigma}})\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}}))^{+}\|)\right] \leq 1 \\
& \mathbf{D} \succeq \mathrm{Diag}(\boldsymbol{r}) \\
& \hat{\boldsymbol{z}} \in \mathbb{R}^L, \ \mathbf{D} \in \mathbb{S}^L_{++}.
\end{aligned}
$$

Subproblems are conic representable!

$$
\begin{aligned}
\inf \quad & \prod_{\ell \in [L]} \frac{\underline{\sigma}_\ell + \bar{\sigma}_\ell}{2} \cdot \det(\mathbf{D}) \\
\text{s.t.} \quad & \mathbb{E}_{\hat{\mathbb{P}}}\left[\psi(\|(\mathrm{Diag}^{-1}(\underline{\boldsymbol{\sigma}})\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}}))^{-} + (\mathrm{Diag}^{-1}(\bar{\boldsymbol{\sigma}})\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}}))^{+}\|)\right] \leq 1 \\
& \underline{\boldsymbol{\sigma}}, \bar{\boldsymbol{\sigma}} \in \mathbb{R}^L_{++}
\end{aligned}
$$

MCVS for Asymmetric Confidence Set



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Predictive Model with Side Information

Consider the data-driven setting. We have a set of historical data with *side information*:

$$\mathcal{D} = \{(\omega, \boldsymbol{z}_\omega, \boldsymbol{s}_\omega)\}_{\omega \in [\Omega]}$$

where $\boldsymbol{z}_\omega \in \mathbb{R}^L$ is the observed outcome of the input vector in scenario ω, and $\boldsymbol{s}_\omega \in \mathbb{R}^L$ is the corresponding *side information*.

Predictive Model with Side Information

We focus on affine prediction map $\boldsymbol{\chi}: \mathbb{R}^S \to \mathbb{R}^L$,

$$\chi(\boldsymbol{s}) = \widehat{\boldsymbol{z}}_0 + \widehat{\mathbf{Z}}\boldsymbol{s},$$

where $\hat{\boldsymbol{z}}_0$ and $\hat{\mathbf{Z}}$ are parameters estimated from data.

Predictive Model with Side Information

The robust constraint with *side information*:

$$f(\boldsymbol{x}, \boldsymbol{z}) \leq \tau + \kappa(\Delta(\boldsymbol{z}, \hat{z}_0 + \hat{\mathbf{Z}}\boldsymbol{s}) - \gamma)^+ \quad \forall \boldsymbol{z} \in \mathcal{Z}$$

Given side information $\boldsymbol{s}$, determine decision on $\boldsymbol{x}$.
Example in unit commitment:

- $\boldsymbol{s}$: Weather information, AI or expert predictions
- $\boldsymbol{z}$: Energy demands
- $\boldsymbol{x}$: Production levels of energy

Predictive Model with Side Information

We utilize the empirical distribution:

$$\hat{\mathbb{P}}\left\{(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{s}}) = (\boldsymbol{z}_\omega, \boldsymbol{s}_\omega)\right\} = 1/\Omega \ \forall \omega \in [\Omega].$$

to estimate parameters $\mathbf{D}$, $\hat{\boldsymbol{z}}_0$ and $\hat{\mathbf{Z}}$.

MVCS with side information:

$$\begin{array}{ll} \inf & \det(\mathbf{D}) \\ \text{s.t.} & \mathbb{E}_{\hat{\mathbb{P}}}\left[\psi(\|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}}_0 - \hat{\mathbf{Z}}\tilde{\boldsymbol{s}})\|)\right] \leq 1 \\ & \mathbf{D} \succeq \text{Diag}(\boldsymbol{r}) \\ & \hat{\boldsymbol{z}}_0 \in \mathbb{R}^L, \ \hat{\mathbf{Z}} \in \mathbb{R}^{L \times S}, \ \mathbf{D} \in \mathbb{S}^L_{++}. \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \sup & (\det(\mathbf{Q}))^{1/L} \\ \text{s.t.} & \mathbb{E}_{\hat{\mathbb{P}}}\left[\psi(\|\mathbf{Q}\tilde{\boldsymbol{z}} - \boldsymbol{q} - \mathbf{P}\tilde{\boldsymbol{s}}\|)\right] \leq 1 \\ & \mathbf{Q} \preceq \text{Diag}^{-1}(\boldsymbol{r}) \\ & \boldsymbol{q} \in \mathbb{R}^L, \ \mathbf{P} \in \mathbb{R}^{L \times S}, \ \mathbf{Q} \in \mathbb{S}^L_{++}. \end{array}$$

- Given the optimal solution of RHS, we can get the optimal solution of LHS:

$$\mathbf{D}^\star = \mathbf{Q}^{\star-1}, \ \hat{\boldsymbol{z}}_0^\star = \mathbf{D}^\star \boldsymbol{q}^\star, \ \text{and} \ \hat{\mathbf{Z}}^\star = \mathbf{D}^\star \mathbf{P}^\star.$$

Generalizes Ordinary Least Square (OLS)

- Optimal solution can coincide with OLS:

$$
\begin{array}{ll}
\inf & \det(\mathbf{D}) \\
\text{s.t.} & \mathbb{E}_{\widehat{\mathbb{P}}}\left[(\|\mathbf{D}^{-1}(\tilde{\boldsymbol{z}} - \hat{\boldsymbol{z}}_0 - \hat{\mathbf{Z}}\tilde{\boldsymbol{s}})\|_2)^2\right] \leq 1 \\
& \hat{\boldsymbol{z}}_0 \in \mathbb{R}^L,\ \hat{\mathbf{Z}} \in \mathbb{R}^{L\times S},\ \mathbf{D} \in \mathbb{S}^L_{++}.
\end{array}
$$

- Statistical properties yet to be discovered.

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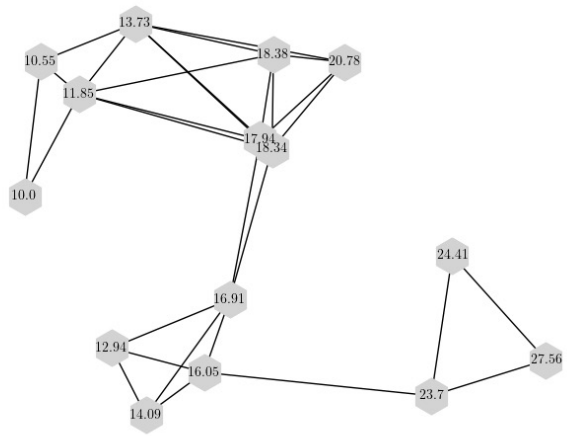
Resource Allocation for Emergency Service

- L nodes with uncertain demands z_ℓ, $\ell \in [L]$
- Here-and-now decision: x_ℓ inventory at location $\ell \in [L]$
- Location ℓ is connected to nodes in $\mathcal{I}_\ell \subseteq [L]$.



Attribute function: Largest demand shortfall

$$f(\boldsymbol{x}, \boldsymbol{z}) = \begin{cases} \min & v \\ \text{s.t.} & \sum_{j \in \mathcal{I}_\ell} y_{\ell j} \leq x_i & \forall \ell \in [L], \\ & \sum_{i \in \mathcal{I}_\ell} y_{i\ell} \geq z_\ell - v & \forall \ell \in [L] \\ & y_{\ell j} \geq 0 & \forall j \in \mathcal{I}_\ell,\ \ell \in [L] \\ & v \in \mathbb{R}. \end{cases}$$

Demands are fulfilled if and only if $f(\boldsymbol{x}, \boldsymbol{z}) \leq 0$.

Empirical Optimization

Resource allocation that covers all historical demands

$$V_{\mathsf{EO}} = \left\{ \begin{array}{ll} \min & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} & f(\boldsymbol{x}, \boldsymbol{z}_\omega) \le 0 \quad \forall \omega \in [\Omega] \\ & \boldsymbol{x} \in \mathcal{X}. \end{array} \right.$$

$$= \left\{ \begin{array}{lll} \min & \boldsymbol{c}^\top \boldsymbol{x} & \\ \text{s.t.} & \sum_{j \in \mathcal{I}_\ell} y_{\ell j \omega} \le x_i & \forall \ell \in [L],\ \omega \in [\Omega] \\ & \sum_{i \in \mathcal{I}_\ell} y_{i \ell \omega} \ge z_{\ell \omega} & \forall \ell \in [L],\ \omega \in [\Omega] \\ & y_{\ell j \omega} \ge 0 & \forall j \in \mathcal{I}_\ell,\ \ell \in [L],\ \omega \in [\Omega] \\ & \boldsymbol{x} \in \mathcal{X}. & \end{array} \right.$$

Budgeted Robust Optimization Model

$$\left\{\begin{array}{lll} \min & \boldsymbol{c}^\top \boldsymbol{x} & \\ \text{s.t.} & f(\boldsymbol{x}, \boldsymbol{z}) \le 0 & \forall \boldsymbol{z} \in \mathcal{Z}(\gamma) \\ & \boldsymbol{x} \in \mathcal{X}. & \end{array}\right.$$

Lifted uncertainty set:

$$\bar{\mathcal{Z}}(\gamma) = \left\{ (\boldsymbol{z}, \underline{\boldsymbol{w}}, \bar{\boldsymbol{w}}, u) \in \mathcal{Z} \times \mathbb{R}^L_+ \times \mathbb{R}^L_+ \times \mathbb{R}_+ \;\middle|\; \begin{array}{l} \bar{\boldsymbol{w}} - \underline{\boldsymbol{w}} = \mathrm{D}^{-1}(\boldsymbol{z} - \hat{\boldsymbol{z}}) \\ \|\underline{\boldsymbol{w}} + \bar{\boldsymbol{w}}\| \le u \le \gamma \end{array} \right\}.$$

- $\mathcal{A}(2L+1)$ is the set of all affine functions:
  - Affine recourse adaption on the on lifted variables

Tolerated Robust Satisficing Model

$$\begin{cases} \min & \kappa \\ \text{s.t.} & f(\boldsymbol{x}, \boldsymbol{z}) \leq \kappa \Delta(\boldsymbol{z}, \hat{\boldsymbol{z}}) \quad \forall \boldsymbol{z} \in \mathcal{Z} \\ & \boldsymbol{c}^\top \boldsymbol{x} \leq V_1 \\ & \boldsymbol{x} \in \mathcal{X}, \kappa \geq 0 \end{cases}$$

RSOME Code

$$\bar{\mathcal{Z}} = \left\{ (\boldsymbol{z}, \underline{\boldsymbol{w}}, \bar{\boldsymbol{w}}, u) \in \mathcal{Z} \times \mathbb{R}^L_+ \times \mathbb{R}^L_+ \times \mathbb{R}_+ \;\middle|\; \begin{array}{l} \bar{\boldsymbol{w}} - \underline{\boldsymbol{w}} = \mathrm{D}^{-1}(\boldsymbol{z} - \hat{\boldsymbol{z}}) \\ \|_{p1} \underline{\boldsymbol{w}} + \bar{\boldsymbol{w}}\| \le u \le \bar{u} \end{array} \right\}.$$

$$\begin{array}{lll}
\min & \kappa & \\
\text{s.t.} & \displaystyle\sum_{j \in \mathcal{I}_\ell} y_{\ell j}(\underline{\boldsymbol{w}}, \bar{\boldsymbol{w}}, u) \le x_\ell & \forall \ell \in [L],\ (\boldsymbol{z}, \underline{\boldsymbol{w}}, \bar{\boldsymbol{w}}, u) \in \bar{\mathcal{Z}} \\
& \kappa u \ge z_\ell - \displaystyle\sum_{i \in \mathcal{I}_\ell} y_{i\ell}(\underline{\boldsymbol{w}}, \bar{\boldsymbol{w}}, u) & \forall \ell \in [L],\ (\boldsymbol{z}, \underline{\boldsymbol{w}}, \bar{\boldsymbol{w}}, u) \in \bar{\mathcal{Z}} \\
& y_{\ell j}(\underline{\boldsymbol{w}}, \bar{\boldsymbol{w}}, u) \ge 0 & \forall j \in \mathcal{I}_\ell,\ \ell \in [L],\ (\boldsymbol{z}, \underline{\boldsymbol{w}}, \bar{\boldsymbol{w}}, u) \in \bar{\mathcal{Z}} \\
& y_{\ell j} \in \mathcal{A}(2L+1) & \forall j \in \mathcal{I}_\ell,\ \ell \in [L] \\
& \boldsymbol{c}^\top \boldsymbol{x} \le V_1 & \\
& \boldsymbol{x} \in \mathcal{X}, \kappa \ge 0. &
\end{array}$$

```python
import numpy as np
import rsome as rso
from rsome import ro
from rsome import cpt_solver as cpt

def rs_decision(c, arcs, xbar, tau, zin, p1=2, p2=2, display=True):

    I = len(c)
    ij= np.array([(i, i) for i in range(I)] +
                 arcs + [a[::-1] for a in arcs])
    K = ij.shape[0]
    q, Q = mvcs(zin, p1, p2, display)
    gbar = np.linalg.norm(zin@Q - q, p1, axis=1).max()

    model = ro.Model()
    z = model.rvar(I)
    w1 = model.rvar(I)
    w2 = model.rvar(I)
    u = model.rvar()
    zset = (w1 - w2 == Q@z - q,
            rso.norm(w1 + w2, p1) <= u, u <= gbar,
            w1 >= 0, w2 >= 0,
            z >= 0, z <= 100)

    x = model.dvar(I)
    kappa = model.dvar()
    y = model.ldr(K)
    y.adapt(w1)
    y.adapt(w2)
    y.adapt(u)

    model.minmax(kappa, zset)
    model.st(c @ x <= tau)
    for i in range(I):
        model.st(y[ij[:, 0] == i].sum() <= x[i])
        model.st(z[i] - y[ij[:, 1] == i].sum() <= kappa * u)
    model.st(x >= 0, x <= xbar, y >= 0)
    model.solve(cpt, display=display)

    objval, xs = model.get(), x.get()
    return objval, xs
```

Out of Sample Evaluations

Comparison results with different norms.

- Average total violation:
  the expected number of unfulfilled demands.

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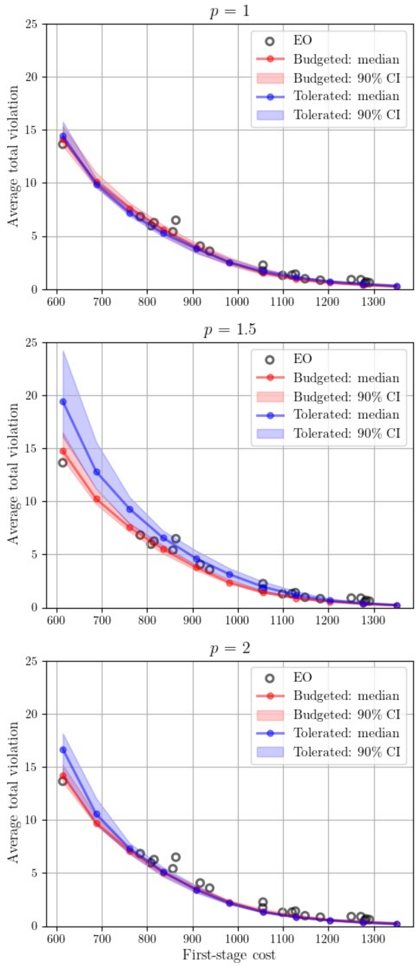

References

- Lou, Z., Chen, Z., Sim, M., Xie, J., & Xiong, P. (2024). Estimation and Prediction Procedures for Unified Robust Decision Models. *Available at SSRN 4890089*.

Thank You！

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