Conditional extragradient algorithms for variational inequalities

Workshop on Metric Bounds and Transversality (WoMBaT)

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Notations

1. $\mathbb{R}^n$: the euclidian n dimensional space
2. $\langle \cdot, \cdot \rangle$: the inner product
3. $|| \cdot ||$: the induced norm
4. $\mathrm{dom}\,(T)$: the domain of T
5. $Gr(T)$: the graph of T
6. $\mathcal{N}_C$: the normal cone of C
7. P_C : the orthogonal projection onto C

Definitions

Orthogonal Projection

By $P_C(x)$ we denote the unique point in a convex, closed and nonempty set C, such that $\|P_C(x) - x\| \leq \|y - x\|$ for all $y \in C$.

Domain and Graph of T

- $\text{dom}\,(T) := \{x \in \mathbb{R}^n \ : \ T(x) \neq \emptyset\}$.
- $Gr(T) := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ u = T(x)\}$.

Normal Cone

Let C be a closed subset of $\mathbb{R}^n$ and let $x \in C$. A vector $u \in \mathbb{R}^n$ is called a *normal* to C at x if for all $y \in C$, $\langle u, y - x\rangle \leq 0$. The collection of all such normal u is called the *normal cone* of C at x and is denoted by $\mathcal{N}_C(x)$. If $x \notin C$, we define $\mathcal{N}_C(x) = \emptyset$.

The Problem and Assumptions

Given a point-to-point operator $T : \mathrm{dom}\,(T) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and a nonempty closed and convex set C, we define:

The Variational Inequality Problem for T and C (VIP(T,C))

$$\text{Find } \; x_* \in C \; \text{ such that } \; \langle T(x_*), x - x_* \rangle \geq 0, \quad \forall x \in C.$$

The Dual Variational Inequality Problem for T and C (DVIP(T,C))

$$\text{Find } \; x_* \in C \; \text{ such that } \; \langle T(x), x - x_* \rangle \geq 0.$$

We denote the solution sets of the**VIP(T,C)** by S_* and the **DVIP(T,C)** by S_0.

Assumptions

(A1) T is continuous on C.
(A2) $\emptyset \neq S_* = S_0$

Some known results

Relationships between S_0 and S_*

- Let $T : \mathrm{dom}\,(T) \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous operator, then $S_0 \subseteq S_*$.
- If T is pseudo-monotone then $S_* \subseteq S_0$.

Fact on orthogonal projection

Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set, $\forall x, y \in \mathbb{R}^n$ and $z \in C$:

- $\langle x - P_C(x), z - P_C(x)\rangle \leq 0.$

The Separating Hyperplane

- $\forall z \in \mathrm{dom}\,(T)$ we have $S_0 \subseteq H(z) := \{y \in \mathbb{R}^n : \langle T(z), y - z\rangle \leq 0\}$.
- Let $S \neq \emptyset$ closed and convex, $x^0 \notin S$ and $S \subseteq W(x) := \{y \in \mathbb{R}^n : \langle y - x, x^0 - x\rangle \leq 0\}$. Then, $x \in B[\frac{1}{2}(x^0 + \overline{x}), \frac{1}{2}\rho]$, where $\overline{x} = P_S(x^0)$ and $\rho = \mathrm{dist}(x^0, S)$.

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The Classical Extragradient Algorithm

Extragradient Algorithm

Step 0 (Initialization): Take $x^0 \in C$.

Step 1 (Iterative Step): Compute

$$z^k = P_C(x^k - \beta_k T(x^k)),$$

$$y^k = \alpha_k z^k + (1 - \alpha_k)x^k,$$

$$\text{and} \quad x^{k+1} = P_C(x^k - \gamma_k T(y^k)).$$

Step 2 (Stopping Test): If $x^{k+1} = x^k$, then stop.

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(a) Constant stepsizes: $\beta_k = \gamma_k = \beta$ where and $\alpha_k = 1$.

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Step 2 (Stopping Test): If $x^{k+1} = x^k$, then stop.

(a) Constant stepsizes: $\beta_k = \gamma_k = \beta$ where and $\alpha_k = 1$.
(b) Linesearch on the boundary of C: Set $\sigma > 0$, and $\delta \in (0, 1)$.
For each k, take $\alpha_k = 1$ and $\beta_k = \sigma 2^{-j(k)}$ where

$$\begin{cases} j(k) := \min\left\{ j \in \mathbb{N} : \|T(x^k) - T(z^{k,j})\| \le \dfrac{\delta}{\sigma 2^{-j}} \|x^k - z^{k,j}\|^2 \right\}, \\ \text{and} \quad z^{k,j} = P_C(x^k - \sigma 2^{-j} T(x^k)). \end{cases}$$

Take $\gamma_k = \beta_k$ and $\gamma_k = \dfrac{\langle T(y^k), x^k - y^k \rangle}{\|T(y^k)\|^2}$.

The Classical Extragradient Algorithm

Extragradient Algorithm

Step 0 (Initialization): Take $x^0 \in C$.

Step 1 (Iterative Step): Compute

$$\begin{aligned} z^k &= P_C(x^k - \beta_k T(x^k)), \\ y^k &= \alpha_k z^k + (1-\alpha_k)x^k, \\ \text{and} \quad x^{k+1} &= P_C(x^k - \gamma_k T(y^k)). \end{aligned}$$

Step 2 (Stopping Test): If $x^{k+1} = x^k$, then stop.

(c) Linesearch along C: Set $\delta \in (0,1)$, $z^k := P_C(x^k - \beta_k T(x^k))$ with, $(\beta_k)_{k\in\mathbb{N}} \subset [\check{\beta}, \hat{\beta}]$ such that $0 < \check{\beta} \le \hat{\beta} < +\infty$, and $\alpha_k = 2^{-\ell(k)}$ where

$$\begin{cases} \ell(k) := \min\left\{\ell \in \mathbb{N} : \langle T(z^{k,\ell}), x^k - z^k\rangle \ge \delta\langle T(x^k), x^k - z^k\rangle\right\}, \\ \text{and} \quad z^{k,\ell} = 2^{-\ell} z^k + (1-2^{-\ell})x^k, \end{cases}$$

Then, define $\gamma_k = \dfrac{\langle T(y^k), x^k - y^k\rangle}{\|T(y^k)\|^2}$.

Main Idea and Example

Extragradient Algorithm with normal vectors

Step 0 (Initialization): Take $x^0 \in C$.

Step 1 (Iterative Step): Compute

$$z^k = P_C(x^k - \beta_k(T(x^k) + u^k)), \quad u^k \in \mathcal{N}_C(x^k)$$

$$y^k = \alpha_k z^k + (1 - \alpha_k)x^k,$$

$$\text{and} \quad x^{k+1} = P_C(x^k - \gamma_k(T(y^k) + v^k)) \quad v^k \in \mathcal{N}_C(y^k).$$

Step 2 (Stopping Test): If $x^{k+1} = x^k$, then stop.

Operator for the Example

Define the operator $\mathcal{R}_{\theta,B}$ by

$$\mathcal{R}_{\theta,B} : \mathbb{R}^2 \to \mathbb{R}^2 : x \mapsto \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}(x - B) + B.$$

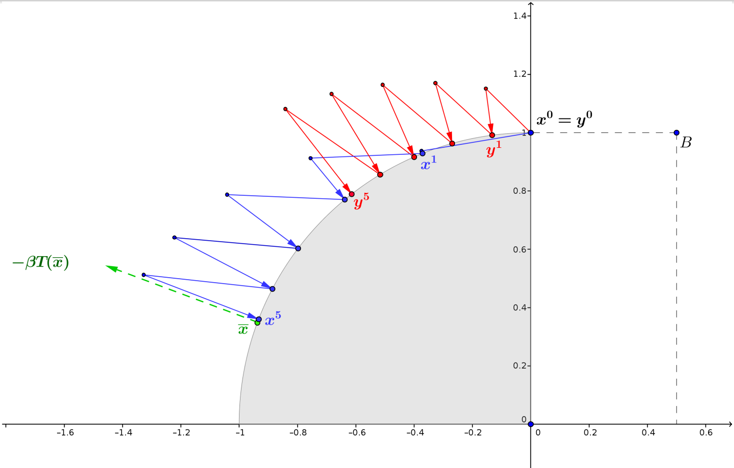
Consider $C := \left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 \le 1,\ x_1 \le 0,\ x_2 \ge 0\right\}$.

Conditional Extragradient with constant stepsizes

Conditional Extragradient Algorithm

Consider $T(x) = \mathcal{R}_{-\frac{\pi}{2},B}x - x$, with $B = (\frac{1}{2}, 1)$, $\beta = 0.25$, $\alpha_k = 1$.

$$z^k = P_C(x^k - \beta(T(x^k) + u^k)), \quad u^k \in \mathcal{N}_C(x^k)$$
$$x^{k+1} = P_C(x^k - \beta(T(z^k) + v^k)) \quad v^k \in \mathcal{N}_C(z^k).$$



Two linesearch with normals

Linesearch B (Linesearch on the boundary)

Input: $(x, u, \sigma, \delta, M)$. $x \in C$, $u \in \mathcal{N}_C(x)$, $\sigma > 0$, $\delta \in (0, 1)$, and $M > 0$.
Set $\alpha = \sigma$ and $\theta \in (0, 1)$ and choose $u \in \mathcal{N}_C(x)$. Denote
$z_\alpha = P_C(x - \alpha(T(x) + \alpha u))$ and choose $v_\alpha \in \mathcal{N}_C(z_\alpha)$ with $\|v_\alpha\| \leq M$.
While $\alpha\|T(z_\alpha) - T(x) + \alpha v_\alpha - \alpha u\| > \delta\|z_\alpha - x\|$ **do**
$\alpha \leftarrow \theta\alpha$ and choose any $v_\alpha \in \mathcal{N}_C(z_\alpha)$ with $\|v_\alpha\| \leq M$.
End While
Output: $(\alpha, z_\alpha, v_\alpha)$.

Two linesearch with normals

Linesearch B (Linesearch on the boundary)

Input: $(x, u, \sigma, \delta, M)$. $x \in C$, $u \in \mathcal{N}_C(x)$, $\sigma > 0$, $\delta \in (0,1)$, and $M > 0$.
Set $\alpha = \sigma$ and $\theta \in (0,1)$ and choose $u \in \mathcal{N}_C(x)$. Denote
$z_\alpha = P_C(x - \alpha(T(x) + \alpha u))$ and choose $v_\alpha \in \mathcal{N}_C(z_\alpha)$ with $\|v_\alpha\| \leq M$.
While $\alpha\|T(z_\alpha) - T(x) + \alpha v_\alpha - \alpha u\| > \delta\|z_\alpha - x\|$ **do**
$\alpha \leftarrow \theta\alpha$ and choose any $v_\alpha \in \mathcal{N}_C(z_\alpha)$ with $\|v_\alpha\| \leq M$.
End While
Output: $(\alpha, z_\alpha, v_\alpha)$.

Linesearch F (Linesearch along the feasible direction)

Input: (x, u, β, δ, M).$x \in C$, $u \in \mathcal{N}_C(x)$, $\beta > 0$, $\delta \in (0,1)$, and $M > 0$.
Set $\alpha \leftarrow 1$ and $\theta \in (0,1)$. Define $z_\alpha = P_C(x - \beta(T(x) + \alpha u))$ and
choose $u \in \mathcal{N}_C(x)$, $v_\alpha \in \mathcal{N}_C(z_\alpha)$ with $\|v_\alpha\| \leq M$.
While $\langle T(\alpha z_\alpha + (1-\alpha)x) + v_\alpha, x - z_\alpha\rangle < \delta\langle T(x) + \alpha u, x - z_\alpha\rangle$ **do**
$\alpha \leftarrow \theta\alpha$ and choose any $v_\alpha \in \mathcal{N}_C(\alpha z_\alpha + (1-\alpha)x)$ with $\|v_\alpha\| \leq M$.
End While
Output: $(\alpha, z_\alpha, v_\alpha)$.

Conceptual Extragradient Algorithm B

Given $\sigma > 0$, $\delta \in (0,1)$, and $M > 0$.

Step 0 (Initialization): Take $x^0 \in C$ and set $k \leftarrow 0$.

Step 1 (Stopping Test 1): If $x^k = P_C(x^k - T(x^k))$, then stop.

Step 2 (Linesearch B): Take $u^k \in \mathcal{N}_C(x^k)$ with $\|u^k\| \le M$ and set

$$(\alpha_k, z^k, v^k) = \textbf{Linesearch B}\ (x^k, u^k, \sigma, \delta, M),$$

i.e., (α_k, z^k, v^k) satisfy
$$\begin{cases} v^k \in \mathcal{N}_C(z^k) \ \text{ with } \ \|v^k\| \le M\ ; \quad \alpha_k \le \sigma; \\ z^k = P_C(x^k - \alpha_k(T(x^k) + \alpha_k u^k)); \\ \alpha_k\|T(z^k) - T(x^k) + \alpha_k(v^k - u^k)\| \le \delta\|z^k - x^k\|. \end{cases}$$

Step 3 (Projection Step): Set $\overline{v}^k := \alpha_k v^k$ and $x^{k+1} := \mathcal{F}_B(x^k)$.

Step 4 (Stopping Test 2): If $x^{k+1} = x^k$ then stop.

$$\mathcal{F}_{B.1}(x^k) = P_C(P_{H(z^k,\overline{v}^k)}(x^k)); \qquad \textbf{(Variant B.1)} \qquad (1)$$

$$\mathcal{F}_{B.2}(x^k) = P_{C\cap H(z^k,\overline{v}^k)}(x^k); \qquad \textbf{(Variant B.2)} \qquad (2)$$

$$\mathcal{F}_{B.3}(x^k) = P_{C\cap H(z^k,\overline{v}^k)\cap W(x^k)}(x^0), \qquad \textbf{(Variant B.3)} \qquad (3)$$

$H(z,v) := \{y : \langle T(z)+v, y-z\rangle \le 0\}$, and $W(x) := \{y : \langle y-x, x^0-x\rangle \le 0\}$.

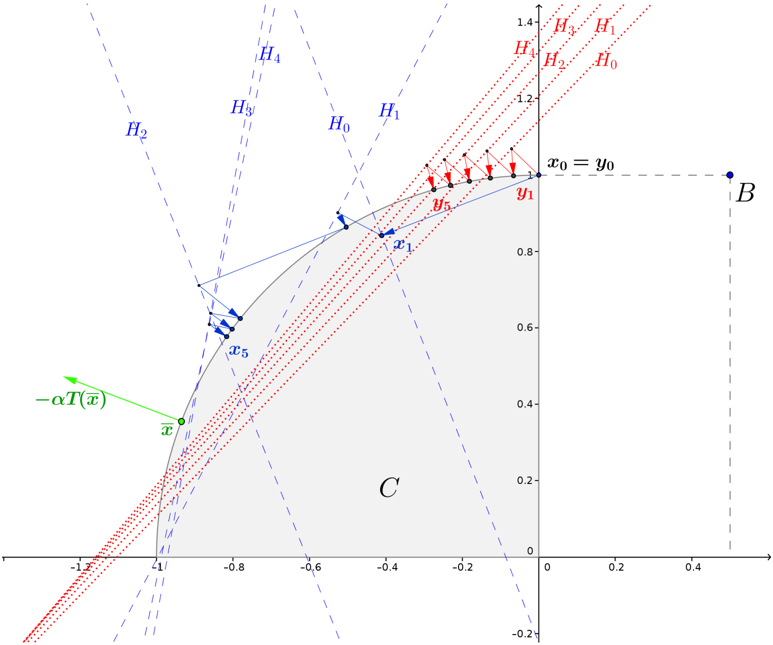


Figure: **Variant B.1**

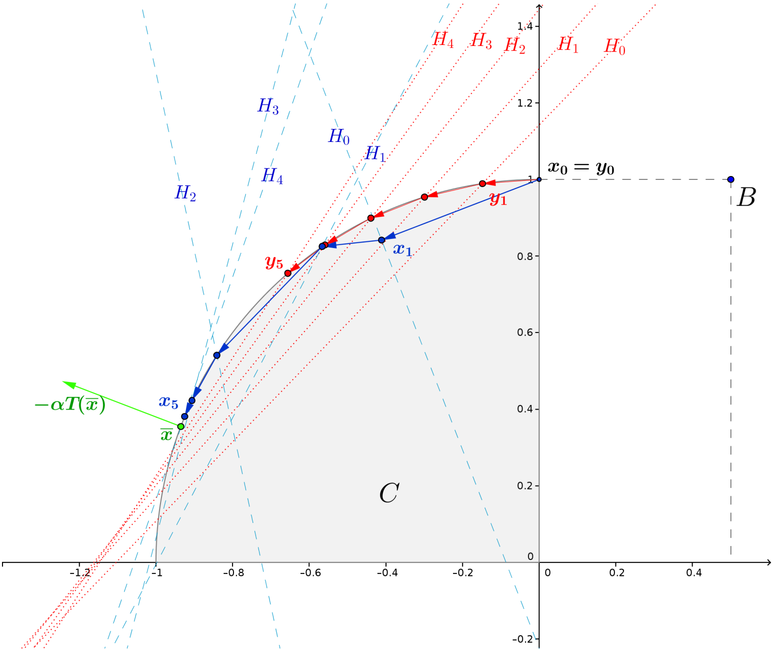


Figure: **Variant B.2**

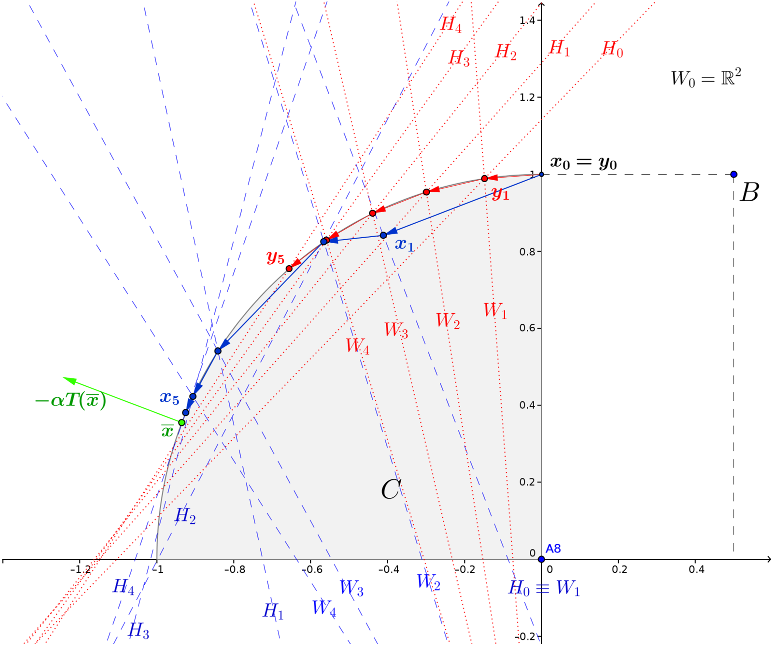


Figure: **Variant B.3**

Conceptual Extragradient Algorithm F

Given $(\beta_k)_{k\in\mathbb{N}} \subset [\check\beta,\hat\beta]$ $0<\check\beta\le\hat\beta<+\infty$, $\delta\in(0,1)$, and $M>0$.

Step 0 (Initialization): Take $x^0\in C$ and set $k\leftarrow 0$.

Step 1 (Linesearch F): Take $u^k\in\mathcal{N}_C(x^k)$ with $\|u^k\|\le M$ and set

$$(\alpha_k, z^k, \bar v^k) = \textbf{Linesearch F}\ (x^k, u^k, \beta_k, \delta, M),$$

i.e., $(\alpha_k, z^k, \bar v^k)$ satisfy

$$\begin{cases} \bar v^k \in \mathcal{N}_C(\alpha_k z^k + (1-\alpha_k)x^k) \text{ with } \|\bar v^k\|\le M\,; \quad \alpha_k\le 1\,; \\ z^k = P_C(x^k - \beta_k(T(x^k)+\alpha_k u^k)); \\ \langle T(\alpha_k z^k + (1-\alpha_k)x^k) + \bar v^k, x^k - z^k\rangle \ge \delta\langle T(x^k)+\alpha_k u^k, x^k - z^k\rangle. \end{cases}$$

Step 2 (Projection Step): Set $\bar x^k := \alpha_k z^k + (1-\alpha_k)x^k$, and $x^{k+1} := \mathcal{F}_F(x^k)$.

Step 3 (Stopping Test 2): If $x^{k+1}=x^k$, then stop.

$$\mathcal{F}_{F.1}(x^k) = P_C(P_{H(\bar x^k,\bar v^k)}(x^k)); \qquad \textbf{(Variant F.1)} \tag{4}$$

$$\mathcal{F}_{F.2}(x^k) = P_{C\cap H(\bar x^k,\bar v^k)}(x^k); \qquad \textbf{(Variant F.2)} \tag{5}$$

$$\mathcal{F}_{F.3}(x^k) = P_{C\cap H(\bar x^k,\bar v^k)\cap W(x^k)}(x^0), \qquad \textbf{(Variant F.3)} \tag{6}$$

$H(z,v) := \{y : \langle T(z)+v, y-z\rangle \le 0\}$, and $W(x) := \{y : \langle y-x, x^0-x\rangle \le 0\}$.

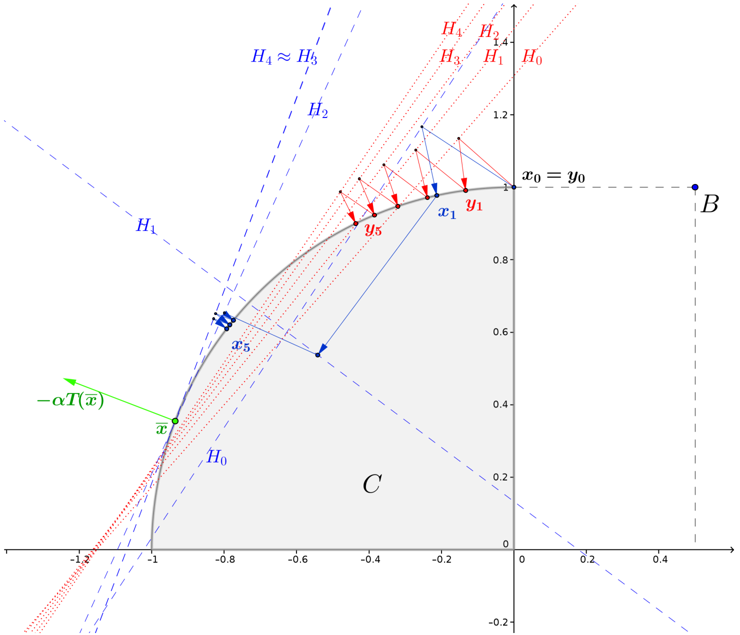


Figure: **Variant F.1**

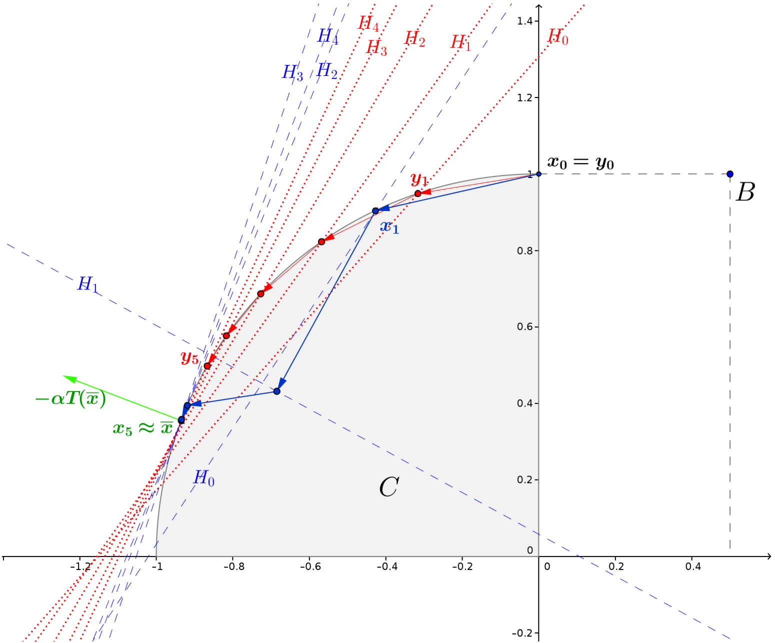


Figure: **Variant F.2**

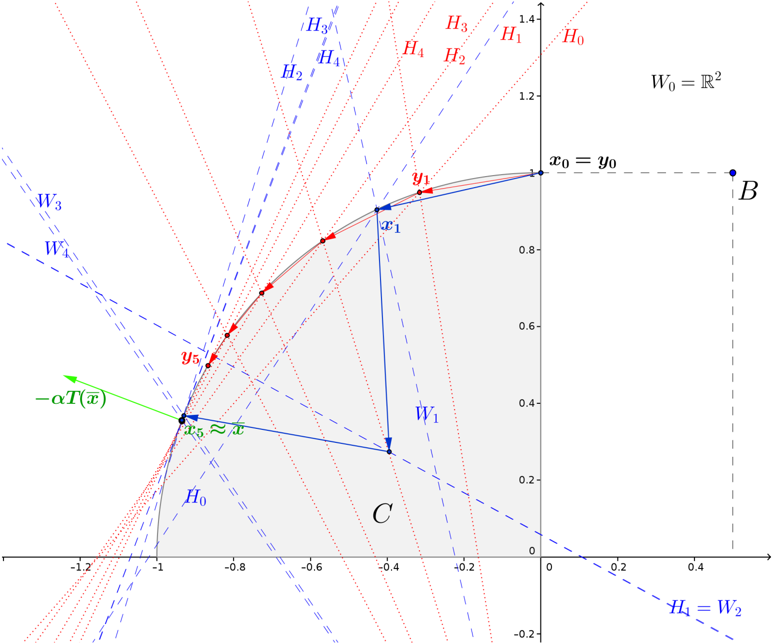


Figure: **Variant F.3**

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Convergence Results

Lemma 1

The sequences generated by **Variants B.1, B.2, F.1** and **F.2** are Fejér convergent to S_*.

Theorem 1

The sequences generated by **Variants B.1, B.2, F.1** and **F.2** converge to a point in S_*.

Lemma 2

The sequences generated by **Variants B.3** and **F.3** belong to

$$B\left[\frac{1}{2}(x^0 + x_*), \frac{1}{2}\mathrm{dist}(x^0, S_*)\right], \qquad x_* = P_{S_*}(x^0).$$

Theorem 2

The sequences generated by **Variants B.3** and **F.3** converge to $x_* = P_{S_*}(x^0)$.

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Thanks!