Turnpike theorems for terminal functionals in infinite horizon

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Outline:

- Turnpike theory
- Continuous time systems
  - Terminal functionals
- Turnpike Theorems

Turnpike Theory

Optimal control problem:

- **System:** $x_{t+1} \in a(x_t), \quad t = 0, 1, 2, \cdots .$
- **Functional** *Maximize:* $\sum_{t=0}^{T} \mathbf{u}$

  where $\mathbf{u} = u(x_t)$ *or* $\mathbf{u} = u(x_t, x_{t+1})$.

Turnpike property describes the "structure/behaviour" of optimal solutions when $T \to \infty$

- $\exists$ "turnpike set/point" that attracts all opt. solutions

- J.V. Neumann, 1932-1945 - first result obtained
  - 1932 - presented at a math.seminar at Princeton (D.Gale)
  - 1937 - published in Vienna
  - 1945 - translated into English
- P.A. Samuelson, 1948-1949 - Interpretation of Neumann's result
- 1958 - the term Turnpike was introduced in
  - *R. Dorfman, P.A. Samuelson and R.M. Solow,* Linear Programming and Economic Analysis, 1958 (Chapter 12)

- A.M. Rubinov, 1973 - Classification of the turnpike property (linear systems - Neumann-Gale model)
  - *V.L. Makarov and A.M. Rubinov,* Mathematical theory of economic dynamics and equilibria, 1973 (Russian)
  - translated into English, 1977
- L. McKenzie, 1976 - Nonlinear systems (bounded trajectories)
  - *L. McKenzie,* Turnpike Theory, Econometrica 44 (1976)

Discrete Systems: the main result

Turnpike property is true for convex problems

($graph\ a$ is convex, u is strongly concave)

Continuous time systems

System: $\dot{x} \in a(x)$

Functional: Utility fun. - $\mathbf{u}(\mathbf{t}) = u(x(t))$ or $u(x(t), \dot{x}(t))$

1. Discounted integral: $\int_0^\infty \mathbf{u}(\mathbf{t})\ e^{-rt} dt$
2. Undiscounted integral: $\int_0^T \mathbf{u}(\mathbf{t})\ dt$
3. Terminal: $\liminf_{t \to \infty} \mathbf{u}(\mathbf{t})$

Main focus: Convex Problems

- $graph\, a = \{(x, y) :\ x \in \Omega,\ y \in a(x)\} \ \Rightarrow$ is convex;
- $u \ \Rightarrow$ is strongly concave.

Turnpike Theorems

Problem (P):

$$\dot{x}(t) \in a(x(t)), \qquad \text{a.e. } t \geq 0; \tag{1}$$

$$\text{Maximize :} \quad J(\mathbf{x}) = \liminf_{t \to \infty} u(x(t)). \tag{2}$$

(i) multi-valued mapping a is defined on convex closed set $\mathcal{D}_a$ with non-empty interior, has compact images and is upper semi-continuous in the Hausdorff metric;

(ii) there exists a bounded solution defined on $[0, \infty)$; that is, the set of solutions denoted by $\mathbf{X} \neq \emptyset$;

(iii) function u is continuous on $\mathcal{D}_u$, where $\mathcal{D}_a \subset \operatorname{int} \mathcal{D}_u$.

- u will be assumed to be quasi-concave or strictly quasi-concave on $\mathcal{D}_u$:

Function u is called quasi-concave if for every $x_1 \neq x_2$

$$u(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u(x_1), u(x_2)\}, \quad \forall \lambda \in (0,1).$$

If the above inequality is strong, $u(x)$ is called strictly quasi-concave.

- The set of stationary points

$$M \triangleq \{x \in \Omega,\ 0 \in \text{co}\, a(x)\}$$

- $x^* \in M$ is optimal stationary point if

$$u(x^*) = \max_{x \in M} u(x)$$

Class of multi-valued mappings $\mathcal{A}$:

given any set $A \subset \mathcal{D}_a$

$$\text{if } 0 \in \operatorname{co} a(A) \quad \text{then} \quad 0 \in \operatorname{co} a(x),\ \exists x \in \operatorname{co} A. \tag{A}$$

Clearly if a has convex images then:

$$0 \in \operatorname{co} a(A) \quad \Rightarrow \quad 0 \in a(\operatorname{co} A). \tag{A^c}$$

The class $\mathcal{A}$ is quite broad.

1: Convex mappings. Denote the graph of mapping a by

$$\operatorname{graph} a \triangleq \{(x,y): \quad x \in \mathcal{D}_a,\ y \in a(x)\}.$$

It is easy to verify that if $\operatorname{graph} a$ is a convex set then condition (A) holds.

Mappings with convex graphs are very important in many applications.

For example, macroeconomic models are usually convex.

2: Linear mappings. Consider linear systems where mapping a is given by

$$a(x) = \{Bx + Cu; \quad u \in U\}.$$

Here B and C are $n \times n$ and $n \times r$ matrices and $U \subset R^r$ is any given set (not necessarily convex). Again, it is not difficult to verify that condition (A) holds without imposing any assumptions on matrices B, C and set U.

Main results

Theorem 0.1: (Upper bound of the functional) Assume that $a \in \mathcal{A}$ and function u is quasi-concave. Then

$$J(\mathbf{x}) \leq u^* \quad \text{for all solutions } \mathbf{x} \in \mathbf{X}. \tag{3}$$

Theorem 0.2: (Turnpike property) Assume that $a \in \mathcal{A}$, function u is strictly quasi-concave and there exists a unique o.s.p. x^*. Then any solution $\mathbf{x} \in \mathbf{X}$ satisfying $J(\mathbf{x}) = u(x^*)$ (i.e. optimal by Theorem 0.1) converges to x^*; that is,

$$\lim_{t \to \infty} x(t) = x^*, \quad \forall \mathbf{x} \in \mathbf{X},\ J(\mathbf{x}) = u(x^*). \tag{4}$$

Corollary 0.3: Assume that $a \in \mathcal{A}$ and M is nonempty, convex and bounded. Then given any strictly quasi-concave function u, there exists a unique o.s.p. x^* and (4) holds.

The proof of Theorem 0.1 is based on the following result:

If $a \in \mathcal{A}$ then

$$\mathrm{co}\, P(\mathbf{x}) \cap M \neq \emptyset, \quad \forall \text{ bounded solutions } \mathbf{x}. \tag{5}$$

Here $P(\mathbf{x})$ is the set of ω-limit points:

$$P(\mathbf{x}) \triangleq \{\xi : \; x(t_k) \to \xi \text{ for some } t_k \to \infty\}.$$

Interesting question: can (5) be satisfied if $a \notin \mathcal{A}$?

- YES in R^2 (we do not have proof for this statement);
- NOT if R^n, $n \geq 3$; that is, relation (5) may not be satisfied for some bounded solution $\mathbf{x}$ if $a \notin \mathcal{A}$.

Example:

$$a(x, y, z) = \{(-y, x, 1 - x^2 - y^2) : \ (x, y, z) \in R^3\}.$$

The solution from an initial point $(1, 0, 0)$ can be obtained as follows:

$$x(t) = \cos(t), \ y(t) = \sin(t), \ z(t) = 0, \quad t \in [0, \infty).$$

This solution is bounded and its ω-limit set is given by

$$P = \{(x, y, z) : \ z = 0, \ x^2 + y^2 = 1\}.$$

It easy to verify that the set

$$\mathrm{co}\, P = \{(x, y, z) : \ z = 0, \ x^2 + y^2 \leq 1\}$$

does not contain any stationary point; that is (5) does not hold.

It can also be shown that $a \notin \mathcal{A}$.

1. Since $a(x,y,z)$ is a singleton $0 \in \operatorname{co} a(x,y,z) \cong 0 = a(x,y,z)$.

Now let $(\dot{x},\dot{y},\dot{z}) = a(x,y,z)$. Clearly, if $x^2+y^2=1$ either $\dot{x} \neq 0$ or $\dot{y} \neq 0$; on the other hand, if $x^2+y^2<1$ then $\dot{z} \neq 0$. Thus, $0 \notin a(x,y,z)$ for all $(x,y,z) \in \operatorname{co} P$.

2. Since the images of a are convex (i.e. singleton) we verify condition (A^c).

Consider the set of two points $A = \{(1,0,0),\ (-1,0,0)\} \subset P$. We have $a(1,0,0) = (0,1,0)$, $a(-1,0,0) = (0,-1,0)$, and therefore

$$(0,0,0) = \frac{1}{2}\,a(1,0,0) + \frac{1}{2}\,a(-1,0,0) \in \operatorname{co} a(A).$$

However, $(0,0,0) \notin a(\operatorname{co} A)$. Indeed, for any $\lambda \in [0,1]$ for the points

$$(x_\lambda, y_\lambda, z_\lambda) = \lambda\,(1,0,0) + (1-\lambda)(-1,0,0) = (2\lambda-1,0,0) \in \operatorname{co} A$$

we have

$$a(x_\lambda, y_\lambda, z_\lambda) = (0, 2\lambda - 1, 1 - (2\lambda - 1)^2) \neq (0,0,0), \quad \forall \lambda \in [0,1]$$

which means that $a \notin \mathcal{A}$.

THANK YOU