# Some Glimpses on Convex Subdifferential Calculus

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# 1. Introduction

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- We provide a general formula for the optimal set of a *relaxed minimization problem* in terms of the *approximate minima* of the data function.

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   a. Subdifferential for the *sum function*.
   b. *Weakening* assumptions ....

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$\theta$ : zero in all the involved spaces.
Given $A, B \subset X$ (or in $X^*$), we consider the operations:

$$A + B := \{a + b \mid a \in A, b \in B\}, \quad A + \emptyset := \emptyset + A := \emptyset;$$

and, if $\Lambda \subset \mathbb{R}$ we set

$$\Lambda A := \{\lambda a \mid \lambda \in \Lambda, \, a \in A\}, \quad \Lambda\emptyset := \emptyset\Lambda = \emptyset.$$

$\mathrm{co}\, A :$ *convex hull* of $A$,

$\mathrm{cone}\, A :$ *conic hull* of $A$ (not convex; i.e. $\mathrm{cone}\, A = \mathbb{R}_+ A$),

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$\mathrm{int}\, A$ : *interior* of $A$,

$\mathrm{cl}\, A$ and $\overline{A}$ : *closure* of $A$ ($w^*$–*closure* if $A \subset X^*$).

$\mathrm{ri}\, A$ : topological *relative interior* of $A$ (i.e., the interior of $A$ in the topology relative to $\mathrm{aff}\, A$ if $\mathrm{aff}\, A$ is closed, and the empty set otherwise).

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$A^\circ := \{x^* \in X^* \mid \langle x, x^* \rangle \geq -1 \; \forall x \in A\}$ : (one-sided) *polar* of $A$.
$A^- := -(\text{cone}\, A)^\circ = \{x^* \in X^* \mid \langle x, x^* \rangle \leq 0 \; \forall x \in A\}$ : *negative dual cone* of $A$.
$A^\perp := (-A^-) \cap A^- = \{x^* \in X^* \mid \langle x, x^* \rangle = 0 \; \forall x \in A\}$ :
*orthogonal subspace* (or *annihilator*) of $A$.

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$\mathrm{N}_A(x) := (A - x)^-$, with $A$ convex and $x \in X$: *normal cone* to $A$ at $x \in A$.
$A_\infty := \{y \in X \mid x + \lambda y \in X \text{ for some } x \in X \text{ and } \forall \lambda \geq 0\}$, with $A$ closed and convex: *recession cone* .

Given $h : X \longrightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, its *(effective) domain* , *epigraph* and *graph* are:

$$\begin{aligned}
\operatorname{dom} h &:= \{x \in X \mid h(x) < +\infty\}, \\
\operatorname{epi} h &:= \{(x, \alpha) \in X \times \mathbb{R} \mid h(x) \leq \alpha\}, \\
\operatorname{gph} h &:= \{(x, h(x)) \in X \times \mathbb{R} \mid x \in \operatorname{dom} h\}.
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The *lsc convex hull* of $h$ is the lsc convex function $\overline{\operatorname{co}} h : X \longrightarrow \overline{\mathbb{R}}$ such that

$$\operatorname{epi}(\overline{\operatorname{co}} h) = \overline{\operatorname{co}}\,(\operatorname{epi} h)\,.$$

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$\Gamma(X)$ : proper lsc convex functions.

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We have $\{h \in \overline{\mathbb{R}}^X : \ h = h^{**}\} = \Gamma(X) \cup \{+\infty\}^X \cup \{-\infty\}^X$. Moreover, $h^{**} \leq \overline{\text{co}}h$, and the equality holds if $h$ admits a continuous affine minorant.

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$\partial_\varepsilon h(x) := \{x^* \in X^* \mid h(y) - h(x) \geq \langle y - x, x^*\rangle - \varepsilon \ \forall y \in X\}$ : *$\varepsilon$−subdifferential* of $h$ at $x \in h^{-1}(\mathbb{R})$ ($\varepsilon \geq 0$).

$\partial_\varepsilon h(x)$ is a $w^*$–closed convex set, and if $h$ is convex, then

$$\partial_\varepsilon h(x) \neq \emptyset \ \forall \varepsilon > 0 \iff h \text{ is lsc at } x.$$

# 3. Optimal set for the *relaxed problem*

Let $h : X \to \overline{\mathbb{R}}$. The *relaxed problem* associated with

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## Theorem 1

*For any function* $h : X \to \overline{\mathbb{R}}$ *such that* $\operatorname{dom} h^* \neq \emptyset$, *one has*

$$\operatorname{argmin}\ h^{**} = \bigcap_{\substack{\varepsilon>0 \\ x^* \in \operatorname{dom} h^*}} \overline{\operatorname{co}}\left( (\varepsilon - \operatorname{argmin} h) + \{x^*\}^- \right).$$

*If* $\operatorname{cone}(\operatorname{dom} h^*)$ *is* $w^*$*−closed or* $\operatorname{ri}(\operatorname{cone}(\operatorname{dom} h^*)) \neq \emptyset$, *then*

$$\operatorname{argmin}\ h^{**} = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left( (\varepsilon - \operatorname{argmin} h) + (\operatorname{dom} h^*)^- \right).$$

*In particular, if* $\operatorname{cone}(\operatorname{dom} h^*)) = X^*$ , *then*

$$\operatorname{argmin}\ h^{**} = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left( \varepsilon - \operatorname{argmin} h \right).$$

Now we proceed with a relevant application of Theorem 1 to the subdifferential calculus.

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**Theorem 2**

*For any function* $h : X \to \overline{\mathbb{R}}$ *such that* $\operatorname{dom} h^* \neq \emptyset$, *one has for all* $x^* \in X^*$,

$$\partial h^*(x^*) = \bigcap_{\substack{\varepsilon>0 \\ u^* \in \operatorname{dom} h^*}} \overline{\operatorname{co}} \left( (\partial_\varepsilon h)^{-1}(x^*) + \{u^* - x^*\}^- \right).$$

*If* $\operatorname{cone}((\operatorname{dom} h^*) - x^*)$ *is* $w^*$*−closed or* $\operatorname{ri}(\operatorname{cone}((\operatorname{dom} h^*) - x^*)) \neq \emptyset$, *then*

$$\partial h^*(x^*) = \bigcap_{\varepsilon>0} \overline{\operatorname{co}} \left( (\partial_\varepsilon h)^{-1}(x^*) + \mathrm{N}_{\operatorname{dom} h^*}(x^*) \right).$$

# 4. Subdifferential of the supremum function

## Theorem 3

*Given* $\{f_t,\ t \in T\} \subset \overline{\mathbb{R}}^X$, $T \neq \emptyset$, *consider the supremum function* $f := \sup_{t\in T} f_t$. *Assume that* $\operatorname{dom} f \neq \emptyset$ *and that*

$$f^{**} \equiv \left(\sup_{t\in T} f_t\right)^{**} = \sup_{t\in T} f_t^{**}. \tag{CC}$$

*Then, at every* $x \in X$, *we have*

$$\partial f(x) = \bigcap_{\varepsilon>0,\ z\in\operatorname{dom} f} \overline{\mathrm{co}}\left(\bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \{z-x\}^-\right),$$

*where* $T_\varepsilon(x) := \{t \in T :\ f_t(x) \geq f(x) - \varepsilon\}$ *if* $f(x) \in \mathbb{R}$ *and* $T_\varepsilon(x) = \emptyset$ *if* $f(x) \notin \mathbb{R}$.

## Theorem 3

*If, moreover,* $\operatorname{cone}\operatorname{co}(\operatorname{dom} f - x)$ *is closed or* $\operatorname{ri}(\operatorname{cone}\operatorname{co}(\operatorname{dom} f - x)) \neq \emptyset$*, then*

$$\partial f(x) = \bigcap_{\varepsilon>0} \overline{\operatorname{co}} \left( \bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \mathrm{N}_{\operatorname{dom} f}(x) \right).$$

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Next we present alternative characterizations of $\mathrm{N}_{\operatorname{dom} f}(x)$ :

$$\begin{aligned} x^* \in \mathrm{N}_{\operatorname{dom} f}(x) &\iff (x^*, \langle x^*, x \rangle) \in [\overline{\operatorname{co}} \, (\cup_{t \in T} \operatorname{gph} f_t^*)]_\infty \\ &\iff (x^*, \langle x^*, z \rangle) \in [\overline{\operatorname{co}} \, (\cup_{t \in T} \operatorname{epi} f_t^*)]_\infty \\ &\iff (x^*, \langle x^*, z \rangle) \in (\operatorname{epi} f^*)_\infty \\ &\iff (x^*, \langle x^*, z \rangle) \in \operatorname{epi}(\sigma_{\operatorname{dom} f}). \end{aligned}$$

Let us define now

$$\mathcal{F}_{x^*} := \left\{ L \subset X^* \;\middle|\; \begin{array}{c} L \text{ is a finite-dimensional linear subspace} \\ \text{such that } x^* \in L \end{array} \right\},$$

## Theorem 5 (HaLoZa'08)

*Given nonempty family* $\{f_t,\ t \in T\} \subset \overline{\mathbb{R}}^X$*, consider the supremum function* $f := \sup_{t \in T} f_t$*, and assume that* $\operatorname{dom} f \neq \emptyset$ *and condition (CC) holds, i.e.*

$$f^{**} = \sup_{t \in T} f_t^{**}.$$

*Then, for every* $x \in X$*,*

$$\partial f(x) = \bigcap_{\varepsilon > 0,\ L \in \mathcal{F}_x} \operatorname{cl}\left(\operatorname{co}\left(\bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x)\right) + \mathrm{N}_{L \cap \operatorname{dom} f}(x)\right).$$

## Corollary 1

*Assume that* $T \neq \emptyset$ *and* $f(x) := \sup\{\langle a_t^*, x\rangle - \beta_t \mid t \in T\}$, *with* $a_t^* \in X^*$ *and* $\beta_t \in \mathbb{R}$. *Then, for every* $x \in X$ *we have*

$$\partial f(x) = \bigcap_{L \in \mathcal{F}_x, \varepsilon > 0} \mathrm{cl}\left(\mathrm{co}\{a_t^* \mid t \in T_\varepsilon(x)\} + B_L\right),$$

*where*

$$x^* \in B_L \Leftrightarrow (x^*, \langle x^*, x\rangle) \in \left[\overline{\mathrm{co}}\left((L^\perp \times \{0\}) \cup \{(a_t^*, \beta_t), t \in T\}\right)\right]_\infty.$$

## Corollary 2

*Let* $\{f_t : X \to \overline{\mathbb{R}} \mid t \in T\}$ *be a non-empty family of convex functions and set* $f := \sup_{t\in T} f_t$*. Assume that one of the following conditions holds:*
*(1) - All the functions* $f_t$ *with* $t \in T$ *are lsc.*
*(2) -* $\exists x_0 \in \operatorname{dom} f$ *such that* $f_t$ *is continuous at* $x_0$, $\forall t \in T$.
*(3) -* $T := \{1, \ldots, k, k+1\}$ *and* $\exists x_0 \in \operatorname{dom} f_{k+1} \cap (\bigcap_{i=1}^{k} \operatorname{dom} f_i)$ *such that* $f_1, \ldots, f_k$ *are continuous at* $x_0$.
*(4) -* $X = \mathbb{R}^n$ *and* $\operatorname{dom} f \cap (\cap_{t\in T} \operatorname{ri}(\operatorname{dom} f_t))$ *is nonempty.*
*Then, (CC) holds and for every* $x \in X$

$$\partial f(x) = \bigcap_{L\in\mathcal{F}_x,\ \varepsilon>0} \operatorname{cl}\left(\operatorname{co}\left(\bigcup_{t\in T_\varepsilon(x)} \partial_\varepsilon f_t(x)\right) + \mathrm{N}_{L\cap \operatorname{dom} f}(x)\right).$$

## Theorem

*Let* $\{f_t : X \to \overline{\mathbb{R}} \mid t \in T\}$ *be a non-empty family of convex functions, and set* $f := \sup_{t\in T} f_t$. *Assume that* $\mathrm{ri}(\mathrm{dom} f) \neq \emptyset$. *Then, we have*

$$\partial f(z) = \bigcap_{\varepsilon>0} \mathrm{cl}\left(\overline{\mathrm{co}}\left(\bigcup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right) + \mathrm{N}_{\mathrm{dom} f}(z)\right).$$

## Theorem (Volle [22], for normed spaces)

*If, additionally,* $f$ *is finite and continuous at* $z \in X$, *then*

$$\partial f(z) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left(\bigcup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right).$$

**Proof.** $f$ is finite and continuous at $z$ and so, $z \in \mathrm{int}(\mathrm{dom} f)$, entailing $\mathrm{N}_{\mathrm{dom} f}(z) = \{\theta\}$. Further, as $z \in \bigcap_{t\in T} \mathrm{int}(\mathrm{dom} f_t)$, condition (2) of Corollary 2 yields $\bar{f} = \sup_{t\in T} \bar{f}_t$, and so the conclusion follows. ■

# 4. Other calculus rules

- $X$ (separated) real locally convex space.
- $f, g : X \to \overline{\mathbb{R}}$ convex functions.

Our formula for the subdifferential of the supremum also yields calculus rules for some other operations, as the sum $g + f$.

**Theorem**

*Assume that the following holds*

$$\overline{g+f} = \overline{g} + \overline{f}.$$

*Then, for every $z \in X$ we have*

$$\partial(g+f)(z) = \bigcap_{\varepsilon>0} \operatorname{cl}\left(\partial_\varepsilon g(z) + \partial_\varepsilon f(z)\right).$$

If $f$ and $g$ are lsc we recover the Hiriart-Urruty & Phelps formula.

Now we recover the Moreau-Rockafellar result:

**Theorem**

*Let $f, g : X \to \overline{\mathbb{R}}$ be convex functions. Assume that $f$ is finite and continuous at $x_0$ for some $x_0 \in \operatorname{dom} g$ . Then*

$$\overline{g+f} = \bar{g} + \bar{f},$$

*and*

$$\partial(f+g)(z) = \partial f(z) + \partial g(z).$$

## Weakening assumptions ....

Consider two convex functions $f, g : X \to \mathbb{R} \cup \{\pm\infty\}$.

1) If $X$ is finite-dimensional and

$$\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset, \tag{1}$$

then

$$\partial(f+g) = \partial f + \partial g. \tag{2}$$

2) If $X$ is a (real) separated locally convex space, (2) holds if

$$\operatorname{dom} f \cap \operatorname{cont} g \neq \emptyset \text{ OR } \operatorname{cont} f \cap \operatorname{dom} g \neq \emptyset. \tag{3}$$

3) (2) also follows replacing (1) by

$$g \text{ is polyhedral AND } \operatorname{dom} g \cap \operatorname{ri}(\operatorname{dom} f) \neq \emptyset.$$

4) If $f, g \in \Gamma(X)$, we have

$$\partial(f+g)(x) = \bigcap_{\varepsilon>0} \operatorname{cl}(\partial_\varepsilon f(x) + \partial_\varepsilon g(x)). \tag{4}$$

The following formula uses the weaker condition that the domains of the involved functions overlap *quasi-sufficiently*.

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## Theorem

*Let* $f$ *and* $g$ *be two convex functions defined on* $X$ *and satisfying* $\overline{f+g} = \bar{f} + g$. *Given* $x \in X$ *such that* $g(x) \in \mathbb{R}$, *we assume either*

$(i)$ $\mathbb{R}_+(\operatorname{epi} g - (x, g(x)))$ *is closed or*

$(ii)$ $\operatorname{dom} f \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset$ *and* $g_{|\operatorname{aff}(\operatorname{dom} g)}$ *is continuous on* $\operatorname{ri}(\operatorname{dom} g)$.

*Then*

$$\partial(f+g)(x) = \bigcap_{\varepsilon>0} \operatorname{cl}(\partial_\varepsilon f(x) + \partial g(x)). \tag{5}$$

Conditions (1) $\overline{f+g} = \bar{f} + g$ and (2) $\overline{f+g} = \bar{f} + \bar{g}$ are not comparable in general: On one hand, for the case $f = g$, condition (2) reads $2\bar{f} = \bar{f} + \bar{f}$, which is obviously always true, while the validity of (1) requires the lsc of $f$. On the other hand, (1) and (2) are equivalent whenever function $g$ is lsc.

Conditions (1) $\overline{f+g} = \bar{f} + g$ and (2) $\overline{f+g} = \bar{f} + \bar{g}$ are not comparable in general: On one hand, for the case $f = g$, condition (2) reads $2\bar{f} = \bar{f} + \bar{f}$, which is obviously always true, while the validity of (1) requires the lsc of $f$. On the other hand, (1) and (2) are equivalent whenever function $g$ is lsc.
This other result can be found in [CoHaLo'16].

**Theorem**

*If $\mathrm{ri}(\mathrm{dom}\, f) \cap \mathrm{ri}(\mathrm{dom}\, g) \neq \emptyset$, and $f_{|\mathrm{aff}(\mathrm{dom}\, f)}$ and $g_{|\mathrm{aff}(\mathrm{dom}\, g)}$ are respectively continuous on $\mathrm{ri}(\mathrm{dom}\, f)$ and $\mathrm{ri}(\mathrm{dom}\, g)$, we have, for every $x \in X$,*

$$\partial(f+g)(x) = \mathrm{cl}\left(\partial f(x) + \partial g(x)\right).$$

*In addition, if one of the subdifferential sets of $f$ or $g$ at $x$ is locally compact, then this last formula reduces to the exact rule*

$$\partial(f+g)(x) = \partial f(x) + \partial g(x).$$

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