Error Bounds for Parametric Polynomial Systems with Applications to Higher-Order Stability Analysis and Convergence Rate

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November 2016, WoMBaT 2016

Joint work with B.S. Mordukhovich, T. A. Nghia and T.S. Pham

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For $f : \mathbb{R}^n \to \mathbb{R}$, we consider the following inequality system

$$(S) \quad f(z) \leq 0.$$

- To judge whether x is an approximate solution of (S), we want to know $d(x, [f \leq 0]) := \inf\{\|x - z\| : f(z) \leq 0\}$.
- However, we often measure $[f(x)]_+ := \max\{f(x), 0\}$.
- So, we seek an error bound: there exist $\tau, \delta > 0$ such that

$$d(x, [f \leq 0]) \leq \tau\left([f(x)]_+ + [f(x)]_+^{\delta}\right)$$

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Definition

We say f has a
(1) global error bound with exponent δ if there exist $\tau > 0$ such that

$$d(x, [f \leq 0]) \leq \tau\big([f(x)]_+ + [f(x)]_+^{\delta}\big) \text{ for all } x \in \mathbb{R}^n$$

(2) local error bound with exponent δ around $\overline{x}$ if there exist $\tau, \epsilon > 0$ such that

$$d(x, [f \leq 0]) \leq \tau\big([f(x)]_+ + [f(x)]_+^{\delta}\big) \text{ for all } x \in \mathbb{B}(\overline{x}; \epsilon).$$

If $\delta = 1$ in (1) (resp. (1)), we say f has a Lipschitz type global (resp. local) error bound.

Error bound is useful in

- analyzing the convergence properties of algorithms (e.g. Luo 2000, Fukushima 2005, Attouch etal. 2009, Tseng 2010 and Izmailov & Solodov 2014);
- sensitivity analysis of optimization problem/variational inequality problem (e.g. Jourani 2000, Ye 2002)
- identifying the active constraints (e.g. Facchinei etal. 1998 and Pang 1997)
- studying maximal monotone operator (Borwein & Dutta 2015) and mixed integer programming problem (Stein, 2016)

Some Known Results

- Lipschitz type global error bound holds when f is maximum of finitely many affine functions (Hoffman 1951)
- Global error bound can fail even when f is convex and continuous (e.g. $f(x_1, x_2) = x_1 + \sqrt{x_1^2 + x_2^2}$).
- Has close link with metric sub-regularity and calmness.
- Many further developments (e.g. Aze, Ioffe, Klatte, Kummer, Kruger, Lewis, Li, López, Ng, Ngai, Outrata, Pang, Robinson, Thera, Ye etc...)

Quadratic cases

- Global error bound with exponent $1/2$ holds when f is a convex quadratic function. (Luo and Luo, 1994); extended to convex quadratic system, (Wang and Pang 1994).
- Local error bound with exponent $1/2$ holds when f is a (nonconvex) quadratic function. (Luo and Sturm, 1998).
- Open questions raised by Luo and Sturm: what happens for the case f can be expressed as finitely many (nonconvex) quadratic functions?

Motivating Example: go beyond quadratic

Consider $f(x) = x^2$. Then, $[f \leq 0] = \{0\}$ and so,

$$d(x, [f \leq 0]) = |x| \leq (x^2)^{\frac{1}{2}} = [f(x)]_+^{\frac{1}{2}}.$$

More generally, consider $f(x) = x^d$ with d is an even number. Then,

$$d(x, [f \leq 0]) = |x| \leq (x^d)^{\frac{1}{d}} = [f(x)]_+^{\frac{1}{d}}.$$

Motivating Example: go beyond quadratic (cont.)

Let d be an even number. Consider $f(x) = \max\{f_1(x), \ldots, f_n(x)\}$ where

$$f_1(x) := x_1^d \text{ and } f_i(x) := x_i^d - x_{i-1},\ i = 2, \cdots, n.$$

Then, $[f \le 0] = \{0\}$. Consider $x(t) = (t^{d^{n-1}}, t^{d^{n-2}}, \cdots, t^d, t) \in \mathbb{R}^n$, $t \in (0,1)$. Then

- $d(x(t), [f \le 0]) = O(t)$;
- $[f(x(t))]_+ = f(x(t)) = t^{d^n}$

So,

$$d(x(t), [f \le 0]) = O\left([f(x(t))]_+^{\frac{1}{d^n}}\right).$$

Recent development for polynomial systems

- Global error bound with exponent $\tau_0 = \frac{1}{(d-1)^n+1}$ holds when f is a convex polynomial with degree d on $\mathbb{R}^n$ (L. SIOPT 2010).
- Local error bound with exponent $\tau_1 = \max\left\{\frac{2}{(2d-1)^n+1}, \frac{1}{\beta(n-1)d^n}\right\}$ if f is maximum of finitely many convex polynomials with degree d on $\mathbb{R}^n$, where $\beta(s)$ is the central binomial coefficient $\binom{s}{[s/2]}$ (Borwein, L. & Yao, SIOPT 2014 and Ngai, SIOPT 2015).
- Local error bound with exponent $\tau_2 = \max\left\{\frac{1}{(d+1)(3d)^{n+r}}, \frac{1}{d(6d-3)^{n+r-1}}\right\}$ when f is maximum of r many (nonconvex) polynomials with degree d on $\mathbb{R}^n$ (L., Mordukhovich, Pham, MP, 2014).


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The Infinite Spark of Being

Main Problem

- Can we extend the error bound results to parametric polynomial system:
$$\phi_l(x,y) \le 0 \text{ for all } y \in \Omega, l = 1,\dots,L,$$
where ϕ_l are polynomials on $\mathbb{R}^n \times \mathbb{R}^m$ with degree d and $\Omega := \{y \in \mathbb{R}^m | \ g_i(y) \le 0,\ i = 1,\dots,r;\ h_j(y) = 0,\ j = 1,\dots,s\}$ with g_i and h_j are all polynomials on $\mathbb{R}^n$ with degree d.
- System of this form arises in semi-infinite programming problem and bilevel programming problem with polynomial data .

Why do we care?

- Many important nonlinear conic programs can be covered in this framework:
  - (1) second order cone constraint with polynomial data:

$$\|(f_1(x),\ldots,f_m(x))\| \le f_0(x)$$
$$\iff \sum_{j=1}^{m} y_j f_j(x) - f_0(x) \le 0 \ \text{ for all } \ \|y\|^2 = 1$$

  - (2) polynomial matrix inequality constraint:

$$P(x) \preceq 0 \iff \sum_{i,j=1}^{m} y_i \left(P(x)\right)_{i,j} y_j \le 0 \ \text{ for all } \ \|y\|^2 = 1.$$

Main Results: Assumptions

We suppose that the set of parameters

$$\Omega = \{y \in \mathbb{R}^m | \ g_i(y) \leq 0, \ i = 1, \ldots, r; \ h_j(y) = 0, \ j = 1, \ldots, s\}$$

be bounded and regular.

What is a regular set?

Ω is regular if, for all $y \in \Omega$, the following MFCQ holds:

$$\left.\begin{array}{l} \sum_{i=1}^{r} \lambda_i \nabla g_i(y) + \sum_{j=1}^{s} \kappa_j \nabla h_j(y) = 0, \\ \lambda_i \geq 0, \ \lambda_i g_i(y) = 0, \ \text{ and } \ \kappa_j \in \mathbb{R} \end{array}\right\} \Longrightarrow \lambda_i = 0, \ \kappa_j = 0 \qquad (3.0)$$

Examples of Regular sets

- Discrete set $\Omega := \{1, \ldots, p\}$, $p \in \mathbb{N}$. Write it as

$$\Omega = \{y \in \mathbb{R} |\ h(y) = 0\} \text{ with } h(y) := (y-1)(y-2)\cdots(y-p).$$

  and observe that $\nabla h(y) \neq 0$ for all $y \in \Omega$.

- The algebraic set

$$\Omega := \{y \in \mathbb{R}^m |\ h_1(y) = 0, \ldots, h_s(y) = 0\}, \quad s \leq m,$$

  is regular provided that $\mathrm{rank}\big(\nabla h_1(y), \ldots, \nabla h_s(y)\big) = s$ for all $y \in \Omega$. (e.g. Sphere under l^p-norm where $p > 1$ is an integer).

Error bounds for parametric polynomial system

Let $R(n,d) := d(3d-3)^{n-1}$.

Theorem

Let $\phi(x) = \max_{\substack{y\in\Omega \\ 1\le l\le L}} f_l(x,y)$ where $f_l\colon \mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$, are polynomials of degrees at most d and

$$\Omega = \left\{y\in\mathbb{R}^m \,\middle|\, g_i(y)\le 0,\ i=1,\dots,r;\ h_j(y)=0,\ j=1,\dots,s\right\}$$

is bounded and regular, where g_i, h_j are polynomials on $\mathbb{R}^m$ with degree d. Then, for any $\bar{x}\in\mathbb{R}^n$ there exist constants $c,\varepsilon>0$ such that

$$d(x,[\phi\le 0]) \le c\,[\phi(x)]_+^{\tau} \quad \text{whenever } \|x-\bar{x}\|\le\varepsilon, \tag{3.0}$$

where $\tau = \frac{1}{R(2n+(m+r+s+2)(n+1),\,d+L+1)}$.

Remark: Can be extended to more general cases where Ω is replaced by a set-valued mapping $Y(x)$ and under weaker regularity assumption.

What is behind the proof?

Łojasiewicz gradient inequality and its variants

- (Łojasiewicz inequality) Let f be an analytic function on $\mathbb{R}^n$ with $f(0) = 0$ and $\nabla f(0) = 0$. Then, exists a rational number $\tau \in (0, 1]$ and $c, \delta > 0$ s.t. $\|\nabla f(x)\| \geq c|f(x)|^{1-\tau}$ for all $\|x\| \leq \delta$.
- (Gwoździewicz 1999 and Kollar 2002) In addition, if f is a polynomial with degree d and 0 is a strict local minimizer, then, $\tau = \frac{1}{(d-1)^n+1}$;
- Dropping the strict minimizer assumption in Gwoździewicz's result, we have a new estimate $\tau = R(n, d)^{-1} = \frac{1}{d(3d-3)^{n-1}}$ (Kurdyka 2012, and L., Mordukhovich and Pham 2014).

Outline of the proof

Goal: Error bound for $\phi(x) = \max_{\substack{y\in\Omega \\ 1\le l\le L}} f_l(x,y)$.

- **Case 1:** $L = 1$. Then $\phi(x) = \max\{f(x,y) : g_i(y) \le 0,\ h_j(y) = 0\}$ and its Lagrangian-type function is

$$F(x,y,\mu,\kappa) := -f(x,y) + \sum_{i=1}^{r} \mu_i^2 g_i(y) + \sum_{j=1}^{s} \kappa_j h_j(y)$$

  (1) Reduce the error bound for ϕ to F;
  (2) F is a single polynomial and so, Łojasiewicz's gradient inequality for single polynomial applies.

- **Case 2:** $L \ge 2$. Reduce it to the case $L = 1$ by using

$$\max_{y\in\Omega} \max_{1\le l\le L} f_l(x,y) = \max_{(y,t)\in\Omega\times\{1,\ldots,L\}} \sum_{l=1}^{L} \gamma_l(t) f_l(x,y).$$

where $\gamma_l : \mathbb{R} \to \mathbb{R}$ is the *Lagrange interpolation polynomials*

Polynomial matrix inequalities: regularity free

Let P be an $(m \times m)$ *polynomial matrix* of n variables with degree d, and let $S_{PMI} = \{x : P(x) \preceq 0\}$.

Corollary

For a compact set $K \subset \mathbb{R}^n$ there is $c > 0$ such that

$$\mathrm{dist}(x, S_{PMI}) \leq c \left(\left[\lambda_{\max}(P(x)) \right]_+ \right)^{\tau} \quad \textit{whenever} \quad x \in K,$$

where $\tau = R(2n + (m+1)(n+1), d+3)^{-1}$ and $\lambda_{\max}$ denotes the maximum eigenvalue of a symmetric matrix.

Recall that:

$$P(x) \preceq 0 \Longleftrightarrow \sum_{i,j=1}^{m} y_i \left(P(x)\right)_{i,j} y_j \leq 0 \;\text{ for all }\; \|y\|^2 = 1.$$

Hunting for the true exponent

Example

Let d be an even number. For any $x = (x_1, \ldots, x_n)$ define $A_1(x) := x_1^d$ and then, for any $i = 2, \ldots, n$,

$$A_i(x) := \begin{pmatrix} -1 & x_i^d \\ x_i^d & -x_{i-1} \end{pmatrix},$$

and the polynomial matrix inequality $P(x) := \mathrm{diag}\big(A_1(x), \ldots, A_n(x)\big)$ Then, $S = \{x : P(x) \preceq 0\} = \{0\}$. For $x(t) = (t^{(2d)^{n-1}}, t^{(2d)^{n-2}}, \cdots, t^{2d}, t) \in \mathbb{R}^n$,

- $d(x(t), S) = O(t)$;
- $\lambda_{\max}\big(P(x(t))\big) = t^{d(2d)^{n-1}}$

Thus, $\tau \leq \frac{1}{d(2d)^{n-1}}$ while our estimate gives $\tau = \frac{1}{(d+3)(3d+6)^{4n-1}}$.

Cyclic Projection Algorithm

(1) Initialization: Given $x_0 \in \mathbb{R}^n$ and finite many closed convex sets $C_1, C_2, \cdots, C_L$ in $\mathbb{R}^n$ with $\bigcap_{l=1}^{L} C_i \neq \emptyset$.

(2) Algorithm: The sequence of *cyclic projections*, $(x_k)_{k\in\mathbb{N}}$, is defined by

$$x_1 := P_1 x_0, x_2 := P_2 x_1, \cdots, x_L := P_L x_{L-1}, x_{L+1} := P_1 x_L \ldots$$

where P_l denotes the Euclidean projection to the set C_l.

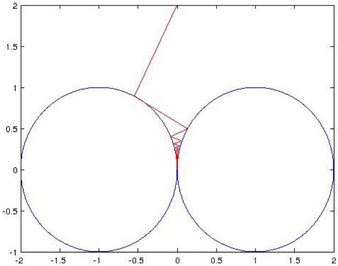
(3) Output: A point in the intersections of C_l.

When $L = 2$, it reduces to alternating projection method.

Let $x_0 = (0, 2)$ and

$$C_1 := \left\{(a, b) \in \mathbb{R}^2 \mid (a+1)^2 + b^2 - 1 \leq 0\right\}$$
$$C_2 := \left\{(a, b) \in \mathbb{R}^2 \mid (a-1)^2 + b^2 - 1 \leq 0\right\}.$$

The following figure depicts the algorithm's trajectory:



This algorithm is easy to implement and was studied by a lot of researchers:

- always convergent to $x_\infty \in \bigcap_{l=1}^{L} C_l$ (Bregman, 1950)
- linearly convergence whenever $\mathrm{int} \bigcap_{l=1}^{L} C_l \neq \emptyset$ (cf. Bauschke & Borwein, 1996).

What is the convergence rate in the degenerate cases e.g. when $\mathrm{int} \bigcap_{l=1}^{L} C_l = \emptyset$?

A partial answer can be given when each C_l has suitable polynomial structures.

Definition

We say C_l, $l = 1, \ldots, L$ are polynomial matrix inequality representable convex sets if C_l is convex and

$$C_l := \left\{x \in \mathbb{R}^n \,\middle|\, A^{(l)}(x) \preceq 0\right\}, \quad l = 1, \ldots, L,$$

where every $A^{(l)} : \mathbb{R}^n \to S^m$ is a polynomial matrix mapping such that each $\left(A^{(l)}(x)\right)_{ij}$ is a real polynomial with degree at most d.

Theorem (L. Mordukhovich, Nghia, Pham 2016)

Let $x_0 \in \mathbb{R}^n$ and let $(x_k)_{k \in \mathbb{N}}$ be the sequence generated by the cyclic projection algorithm for the above C_l. Then $x_k \to x_\infty \in \bigcap_{l=1}^{L} C_l$, and there exists $M > 0$ such that

$$\|x_k - x_\infty\| \leq M \frac{1}{k^\rho}, \quad \forall k \in \mathbb{N},$$

where $\rho := \frac{1}{[2\, R(2n+(m+3)(n+1), d+L) - 2]^{-1}}$.

Further recent advance on cyclic (alternating) projection algorithm

- For the linear matrix inequality case, sharper convergence rate for was obtained in " Drusvyatskiy, L., & Wolkowicz, Alternating projections for ill-posed semi-definite feasibility problems, MP 2016"

Other algorithms

In the semi-algebraic setting, similar techniques can also be used to analyzing the convergence rate of

- proximal point algorithm (Bolte & Attouch, 2013;L. & Mordukhovich, 2012)
- Douglas-Rachford algorithm and one of its variant (L. and Pong, MP, 2016; Borwein, L., Tam, to appear in SIOPT).

High-order stability for polynomial problems

Consider the following parameterized problem:

$$(PMI_u) \qquad \text{minimize} \;\; f(x,u) \;\; \text{subject to} \;\; P(x) \preceq 0,$$

where $u \in \mathbb{R}^l$ is the perturbation parameter, and

- $f(\cdot, u)$ is a polynomial with degree d and $f(x, \cdot)$ is locally Lipschitzian.
- $P : \mathbb{R}^n \to S^m$ is such that each (i,j)th element $\big(P(x)\big)_{ij}$, $1 \leq i,j \leq m$, is a real polynomial with degree d.
- the feasible set is compact.

For each $u \in \mathbb{R}^l$ denote the solution set of $(POP)_u$ by $S(u)$.

Example:

$$\max_{x\in\mathbb{R}^n} \quad f(x,u) := p_0(x) + \sum_{i=1}^{l} u_i p_i(x)$$

$$\text{subject to} \quad A_0 + \sum_{j=1}^{m} x_j A_j + \sum_{j,k=1}^{m} x_j x_k B_{jk} \preceq 0,$$

where all p_i are polynomials.

Hölder continuity of solution maps

Theorem (L., Mordukhovich, Nghia, & Pham, 2016)

For (POP_u), let $\overline{u} \in \mathbb{R}^l$. Then, there are constants $c, \delta > 0$ such that

$$S(u) \subset S(\overline{u}) + c\,\|u - \overline{u}\|^{\tau}\, \overline{\mathbb{B}}(0,1) \quad \textit{whenever} \quad \|u - \overline{u}\| \leq \delta$$

with the explicit exponent

$$\tau = R\big(2n + (m+3)(n+1), d+4\big)^{-1}.$$

- Similar Hölder continuity of solution maps for nonlinear 2nd-order cone program and generalized semi-infinite program with polynomial data were also provided.
- Has important application in deriving high-order semismooth property for maximum eigenvalue of a tensor (L., Qi & Yu 2013 and L., Mordukhovich & Pham 2014).

Conclusion

- Error bound is an interesting research topic and has many important applications;
- Variational analysis and semi-algebraic techniques could shed some light on how to improve error bound results from quadratic to polynomial cases.

Future Works

Still very preliminary development. A lot of interesting questions, e.g.

(1) How to sharpen the derived exponent?

(2) How to estimate/compute the error bound constant c and the radius constant δ (for local cases)?

(3) For the stability result, what happens if we also perturb the constraint functions ?

Want to know more?

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(7) G. Li, L.Q. Qi, G. Yu, Semismoothness of the maximum eigenvalue function of a symmetric tensor and its application. Linear Algebra Appl. 438 (2013), no. 2, 813-833.

Thanks !

$f(x_1, x_2) = x_1 + \sqrt{x_1^2 + x_2^2}$. $[f \leq 0] = \{(x_1, x_2) : x_1 \leq 0, x_2 = 0\}$.
Consider $x^n = (-n, 1)$. Then $d(x^n, [f \leq 0]) = 1$ and
$f(x^n) = -n + \sqrt{n^2 + 1} = \frac{1}{\sqrt{n^2+1}+n} \to 0$.