Tilt Stability Revisited

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WoMBaT
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Outline of Talk

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1. Sufficient Optimality Conditions smooth case
2. A very Nice Class of Problems - Prox-regularity
3. Stable Strong Local Minimizers and Strict local Minimisers order 2
4. Strong Metric Regularity
5. More on Stable Strong Local Minimizers
6. Geometric Description of Subgradients and Second Order Notions
7. Metric Regularity, Pseudo-Lipschitzness and Coderivatives
8. Convexification in Tilt Stability
9. $\mathcal{VU}$ decompositions - the new frontier.

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Sufficient Optimality Conditions

- It is very instructive to first consider smooth optimisation and the second order sufficiency. If f a continuous second partial derivatives the $\bar{x}$ is a local minimum if we have the following conditions satisfied.

$$\nabla f(\bar{x}) = 0 \tag{1}$$

$$d^T \nabla^2 f(\bar{x}) d > 0 \quad \text{for all } \|d\| = 1 \tag{2}$$

- By continuity we know that (2) can be extended to hold on a ball and so for $x \in B_\delta(\bar{x})$ we have $d^T \nabla^2 f(x) d > 0$ giving strict convexity locally.

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Sufficient Optimality Conditions

- Such conditions imply that $x \mapsto f(x)$ is strongly convex on $B_\delta(\bar{x})$ i.e. $f(x) \geqslant f(x') + \langle \nabla f(x'), x - x' \rangle + \frac{\lambda_{\min}}{2} \|x - x'\|^2$.
- So local convexification around a point where second order sufficiency holds is a well known phenomena in smooth optimization.
- We want to remind ourselves as to what conditions replace these in nonsmooth optimisation via a reformulation of the smooth conditions.

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- So local convexification around a point where second order sufficiency holds is a well known phenomena in smooth optimization.
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Sufficient Optimality Conditions

- The strong convexity of $x \mapsto f(x)$ on $B_\delta(\bar{x})$ means that supporting hyperlanes lies below the graph of f.
- More strongly we have for all $x, x' \in B_\delta(\bar{x})$,

$$f(x) - \langle x, \nabla f(x') \rangle \geqslant f(x') - \langle x', \nabla f(x') \rangle + \frac{\lambda_{\min}}{2} \|x' - x\|^2$$

("a strongly stable minima") so for $z' = \nabla f(x')$ we have

$$x' = (\nabla f)^{-1}(z') = \arg\min\{x \mid f(x) - \langle x, z' \rangle\} := m(z').$$

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Sufficient Optimality Conditions

- The existence of the inverse $(\nabla f)^{-1}(z)$ follows from the strong monotonicity of $x \mapsto \nabla f(x)$.
- Taking the inequality

$$f(x) - \langle x, \nabla f(x') \rangle \geqslant f(x') - \langle x', \nabla f(x') \rangle + \frac{\lambda_{\min}}{2} \|x' - x\|^2$$
$$f(x') - \langle x', \nabla f(x) \rangle \geqslant f(x) - \langle x, \nabla f(x) \rangle + \frac{\lambda_{\min}}{2} \|x' - x\|^2$$

twice but swapping the roles of the x and x' then adding gives

$$\left\langle \frac{x' - x}{\|x' - x\|}, \frac{\nabla f(x') - \nabla f(x)}{\|x' - x\|} \right\rangle \geqslant \lambda_{\min}.$$

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Sufficient Optimality Conditions

- Using $z' = \nabla f(x')$ and $z = \nabla f(x)$ we have by definition $x' = m(z')$ and $x = m(z)$ giving (with an application of schwarz inequality)

$$\left(\frac{1}{\lambda_{\min}}\right)\|z' - z\| \geqslant \|m(z') - m(z)\|,$$

a Lipschitz property for $z \mapsto m(z)$.

Definition

A point $\bar{x}$ gives a tilt stable local minimum of a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$ if $f(\bar{x})$ is finite and there exists an $\varepsilon > 0$ such that the mapping

$$m : z \mapsto \arg\min_{\|x - \bar{x}\| \leqslant \varepsilon} \{f(x) - \langle x, z\rangle\} \tag{3}$$

is single valued and Lipschitz on some neighbourhood of 0 with $m(0) = \bar{x}$.

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A very Nice Class of Problems - Prox-regularity

- The purpose of the talk is to discuss how to connect similar notions for the "nicest" non-trivial class of nonsmooth functions that can be considered. These consists of problems that can be modelled as "nonlinear programming problems modeled via nonsmooth analysis".
- We want to see what extra is need to close the loop in these connections.
- Problems like

$$\min f(x) + g(F(x))$$

where F is twice continuously differentiable and g is extended-real-valued convex and f is a nonsmooth objective. We also require a CQ of the form

$$\nexists y \neq 0 \text{ with } y \in N_{\operatorname{dom} g}(F(\bar{x})) \text{ with } \nabla F(\bar{x})^* y = 0.$$

Such functions are in the class of prox-regular, subdifferentially continuous functions.

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Proximal Subdifferentials

- If we lift a quadratic minorant so that it touches the epigraph of $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ at $x' \in \operatorname{dom} f$ we obtain the proximal–subdifferential condition

$$f(x) \geqslant f(x') + \langle z', x - x' \rangle - \frac{r}{2}\|x - x'\|^2. \qquad (4)$$

- Denote by $\partial_p f(x')$ the proximal subdifferential, which consists of all vectors z' satisfying (4) in some neighbourhood of x'.
- f is prox-regular at $\bar{x}$ with respect to $r > 0$ of (4) is satisfied uniformly in r within some ball $x, x' \in B_\delta(\bar{x})$ (for some $\delta > 0$) for all $z' \in \partial f(x')$ for which $f(x)$ is close to $f(x')$.
- We say f is subdifferentially continuous at $\bar{x}$ for $\bar{z} \in \partial f(\bar{x})$ iff $f(x) \to f(\bar{x})$ whenever $(x, z) \to (\bar{x}, \bar{z})$ with $z \in \partial f(x)$.

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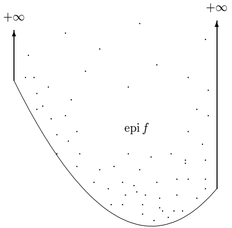
The Limiting Subdifferentials

- The limiting subdifferential is given for all x by

$$\begin{aligned} & \partial f(x) \\ = \quad & \limsup_{x' \xrightarrow{f} x} \partial_p f(x') := \{z \mid \exists z_v \in \partial_p f(x_v), x_v \xrightarrow{f} x, \text{ with } z_v \to z\}, \end{aligned}$$

  where $x' \xrightarrow{f} x$ means that $x' \to x$ and $f(x') \to f(x)$.
- Define epi $f := \{(x, \alpha) \mid \alpha \geqslant f(x)\}$ which is just a set in $\mathbb{R}^{n+1}$



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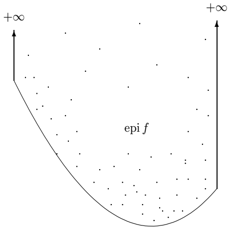

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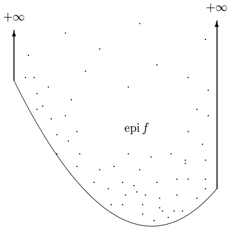

- When we have a a function that is prox-regular and subdifferentially continuous at $(\bar{x}, \bar{z})$ then locally proximal and limiting subgradients coincide.

Stable Strong Local Minimizers and Strict local Minimizers order 2

- One very classical notion is that of a strict local minimum order two at $\bar{x}$ if the there exists $\beta, \delta > 0$ such that

$$f(x) \geqslant f(\bar{x}) + \beta\|x - \bar{x}\|^2 \quad \text{for all } x \in B_\delta(\bar{x}).$$

- When this is applied to the perturbed function $f_z(x) := f(x) - \langle z, x\rangle$ and we demand fixed constants $\beta, \delta > 0$ that apply locally at $(x_z, z) \in \operatorname{Graph} \partial f \cap B_\varepsilon(\bar{x}, 0)$ then we have a strong local minimizer i.e.

$$f_z(x) \geqslant f_z(x_z) + \beta\|x - x_z\|^2 \quad \text{for all } x \in B_\delta(\bar{x}).$$

- This is also referred to as the second-order growth condition.

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Stable Strong Local Minimizers

- The same calculation perform before give the observation that when we have a stable strong local minimizer then we have the mapping $z \mapsto x_z$ is locally Lipschitz (with Lipschitz constant $\frac{1}{2\beta}$) i.e. for all $(x_z, z), (x_v, v) \in \operatorname{Graph} \partial f \cap B_\varepsilon(\bar{x}, 0)$ we have
- Subtracting the two equations

$$\begin{aligned} f(x_z) &\geqslant f(x_v) + \langle v, x_z - x_v \rangle + \beta \|x_z - x_v\|^2 \\ f(x_v) &\geqslant f(x_z) + \langle z, x_v - x_z \rangle + \beta \|x_z - x_v\|^2 \end{aligned}$$

we have $$\frac{\|v - z\|}{\|x_v - x_z\|} \geqslant \langle \frac{v - z}{\|x_v - x_z\|}, \frac{x_v - x_z}{\|x_v - x_z\|} \geqslant 2\beta$$

and so $\|x_v - x_z\| \leqslant \frac{1}{2\beta}\|v - z\|$. Moreover $v \mapsto (\partial f)^{-1}(v) \cap B_\varepsilon(0)$ is single valued. Thus we have a tilt stable minimizer.

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we have
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Strong Metric Regularity

- We now observe that the single valuedness and Lipschitz behaviour of $v \mapsto (\partial f)^{-1}(v) \cap B_\varepsilon(0)$ correspond to the notion of strong metric regularity.
- A multi-function F is metric regular around $(\bar{x}, \bar{y}) \in \operatorname{Graph} F$ iff there exist neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ such that for all $(x, y) \in U \times V$ we have

$$d(x, F^{-1}(y)) \leqslant \kappa d(y, F(x)).$$

- When we fix $y = \bar{y}$ we say that we have the weaker notion of metric sub-regularity.
- If we demand single valuedness of $y \mapsto F^{-1}(y)$ locally then we say that we have a strong version of these properties.
- Clearly we have seen that second-order growth condition force $v \mapsto (\partial f)^{-1}(v) \cap B_\varepsilon(0)$ to be strongly metric regular.

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- A multi-function F is metric regular around $(\bar{x}, \bar{y}) \in \operatorname{Graph} F$ iff there exist neighbourhoods U of $\bar{x}$ and V of $\bar{y}$ such that for all $(x, y) \in U \times V$ we have

$$d(x, F^{-1}(y)) \leqslant \kappa d(y, F(x)).$$

- When we fix $y = \bar{y}$ we say that we have the weaker notion of metric sub-regularity.
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More on Stable Strong Local Minimizers

- Mordukhovich and Nghia first showed that for a lower semi-continuous convex function (on a Banach space) we have the following equivalent:
  1. The subdifferential $\partial f : X \to X^*$ is strongly metric regular around $(\bar{x}, \bar{x}^*)$ with modulus $\kappa > 0$.
  2. There are neighbourhoods U and V of $\bar{x}$ and $\bar{x}^*$, respectively, such that the mapping $(\partial f)^{-1}$ admits a single valued localisation on which we also have the second order growth condition.

  Moreover the following are equivalent:
  1. The point $\bar{x}$ is a global minimizer of f and the subgradient is strongly metric regular around $(\bar{x}, 0)$ with modulus κ
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A Geometric Description of Subgradients

- One can define a normal cones to sets and one of the simplest is following

$$\hat{N}_C(\bar{x}) := \{x \mid \exists \tau > 0 \text{ s.t. } \langle v, x - \bar{x} \rangle \leqslant \frac{1}{2\tau}\|x - \bar{x}\|^2 \text{ for all } x \in C\}.$$

- This motivates one to consider

$$\left\{ v \mid (v, -1) \in \hat{N}_{\mathrm{epi}\, f}(\bar{x}, f(\bar{x})) \right\}$$

and remarkably this set the same as $\partial_p f(\bar{x})$ defined before.

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- This motivates people to apply the last very general construction to *any* multi-function $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$.
- Denote its graph by Graph $F := \{(x, y) \mid y \in F(x)\}$.
- The *Mordukhovich coderivatives* is defined as

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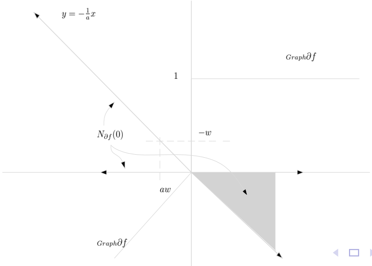

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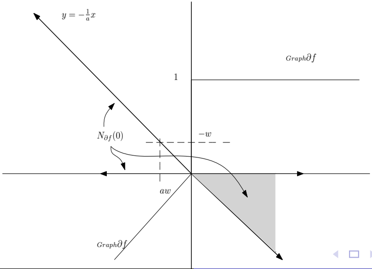

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Metric Regularity, Pseudo-Lipschitzness and Coderivatives

The following are well known to be equivalent:

- A multi-function $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is metrically regular at $(\bar{x}, \bar{z})$.
- There exists $\mu, \eta > 0$ such that for all $(x, z) \in B_\eta(\bar{x}) \times [\Gamma(x) \cap B_\eta(\bar{y})]$ and $w \in D^*\Gamma(x, z)(y)$ we have $\|y\| \leqslant \mu \|w\|$.
- The multi-function $\Gamma^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ has the Aubin (or Pseudo-Lischitzian) property: There exist neighbourhoods of $\bar{x}$ and $\bar{y}$

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Convexification in Tilt Stability

- The convex hull of a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is denoted by $\operatorname{co} f$ and corresponds to the proper lower-semi-continuous function whose epigraph is given by $\overline{\operatorname{co} \operatorname{epi} f}$ (the smallest closed convex set containing the graph).
- From the definition of tilt stability we have on $B_\varepsilon(\bar{x})$ that

$$f(x) \geqslant f(m(v)) + \langle x - m(v), v \rangle$$

(i.e. $v \in \partial_{\text{co}} f(m(v))$) where $m(\cdot)$ is as defined in (3), and hence

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Proposition

Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}_{+\infty}$ *is a proper lower semi-continuous function and suppose that* $m(z) \neq \emptyset$. *Then for all sufficiently small* $\varepsilon > 0$, *in terms of the function* $h(w) := f(\bar{x} + w) + \delta_{B_\varepsilon(0)}(w)$ *we have*

$$\operatorname{co} \arg \min_{x \in B_\varepsilon(\bar{x})} [f(x) - \langle x, z \rangle] = \arg \min_{w' \in \mathbb{R}^n} [\operatorname{co} h(w) - \langle v, z \rangle]$$

for all z *sufficiently close to* 0. *Consequently when* $\bar{x}$ *is a tilt stable local minimum of* f *we have*

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and a tilt stable local minimum of $\operatorname{co} h$ *at* $x = 0$.

Equivalence of Metric Regularity vs Strong Metric Regularity

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be l.s.c. By the necessary\sufficient optimality conditions for convex function we have

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\begin{aligned}
m(z) &:= \arg\min_{x \in B_\varepsilon(\bar{x})} \left[f(x) - \langle x, z \rangle \right] \\
&\subseteq \operatorname{co} \arg\min_{x \in B_\varepsilon(\bar{x})} \left[f(x) - \langle x, z \rangle \right] \\
&= \arg\min_{w' \in \mathbb{R}^n} \left[\operatorname{co} h(w) - \langle v, z \rangle \right] = (\partial \operatorname{co} h)^{-1}(z) \cap B_\varepsilon(\bar{x}) .
\end{aligned}
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- Now as $(\partial \operatorname{co} h)^{-1}(z) = \partial h^*(z)$ we have $m(z)$ single valued and Lipschitz whenever $\partial \operatorname{co} h$ is strongly metrically regular.
- In fact we only require $\partial \operatorname{co} h$ metrically regular for this to occur thanks to the following result.

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$$\begin{aligned} m(z) &:= \arg\min_{x \in B_\varepsilon(\bar{x})} \left[f(x) - \langle x, z \rangle \right] \\ &\subseteq \operatorname{co} \arg\min_{x \in B_\varepsilon(\bar{x})} \left[f(x) - \langle x, z \rangle \right] \\ &= \arg\min_{w' \in \mathbb{R}^n} \left[\operatorname{co} h(w) - \langle v, z \rangle \right] = (\partial \operatorname{co} h)^{-1}(z) \cap B_\varepsilon(\bar{x}) . \end{aligned}$$

- Now as $(\partial \operatorname{co} h)^{-1}(z) = \partial h^*(z)$ we have $m(z)$ single valued and Lipschitz whenever $\partial \operatorname{co} h$ is strongly metrically regular.
- In fact we only require $\partial \operatorname{co} h$ metrically regular for this to occur thanks to the following result.

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Theorem

Let H be a Hilbert space, $f : H \to \mathbb{R}_{+\infty}$ be lsc, prox-regular, and subdifferentially continuous at $\bar{x} \in \operatorname{int}(\operatorname{dom}\partial f)$ for some $\bar{v} \in \partial f(\bar{x})$. Assume in addition that the subdifferential mapping ∂f is pseudo-Lipschitz (or Lipschitz like) with modulus $L \geqslant 0$ around $(\bar{x}, \bar{v})$. Then there exists $\epsilon > 0$ such that $\partial f(x) = \{\nabla f(x)\}$ for all $x \in B_\epsilon(\bar{x})$ with the Lipschitzian derivative $x \mapsto \nabla f(x)$ on $B_\epsilon(\bar{x})$.

- If $\partial \operatorname{co} h$ is metrically regular at $(0, \bar{z})$ then $(\partial \operatorname{co} h)^{-1}(z) = \partial h^*(z)$ must be pseudo-Lipschitz around $(0, \bar{z})$.
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- We see that $\partial \operatorname{co} h$ is metrically regular at $(0, \bar{z})$ iff $\partial \operatorname{co} h$ is strongly metrically regular at $(0, \bar{z})$
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An open question: Suppose f is prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{z} \in \partial f(\bar{x})$. Is ∂f is metrically regular at $(\bar{x}, \bar{z})$ iff ∂f is strongly metrically regular at $(\bar{x}, \bar{z})$?

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Coderivative definiteness and the Mordukhovich criteria

- The the second order sufficiency condition

$$\text{for all } p \in D^* (\partial f)(\bar{x}, 0)(h) \text{ we have } \langle p, h \rangle \geqslant \beta \|h\|^2 > 0 \quad (6)$$

implies

- the sufficient (and necessary condition) for metric regularity i.e.

$$\begin{aligned} \|p\|\|h\| &\geqslant \langle p, h \rangle \geqslant \beta \|h\| \\ \text{or} \quad \frac{1}{\beta}\|p\| &\geqslant \|h\| \quad \text{for all } p \in D^* (\partial f)(\bar{x}, 0)(h). \end{aligned}$$

- and the Modukhovich condition for a Lipschitz like behaviour of $p \mapsto (\partial f)^{-1}(p)$ i.e.

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$$\text{for all } p \in \partial^2 f(\bar{x}, 0)(h) := D^*(\partial f)(\bar{x}, 0)(h) \text{ we have } \langle p, h \rangle \geqslant 0 \quad (7)$$

- But this is not so: Let $f(x_1, x_2) := (x_1^2 - x_2^2) + \delta_\Omega(x_1, x_2)$, where $\Omega := \{(x_1, x_2) \mid (x_1 - x_2, x_1 + x_2) \in \mathbb{R}^2_+\}$. Then one can show $(0, -2) \in \partial^2 f((0, 0), 0)(0, 1)$ and so $\langle (0, -2), (0, 1) \rangle = -2 < 0$.
- This is compounded with the fact that one can replace (6) with the following: There exists $\kappa > 0$ such that for any $r \in [0, \kappa^{-1})$ we have

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- Moreover the quadratic growth condition and its equivalence to tilt stability and the former clearly implies the existence of $\delta > 0$ such that for all $x \in B_\delta(\bar{x})$ we have

$$f''_{-}(x, z, h) \quad := \quad \liminf_{\substack{t \downarrow 0 \\ h' \to h}} \frac{2}{t^2}(f(x + th) - f(x) - \langle z, h \rangle) > 0$$

for all $(x, z) \in \operatorname{Graph} \partial f \cap B_\delta(\bar{x})$ and $\|h\| = 1$.

- This is actually equivalent to the positive definite coderivative condition (6) when f is prox-regular and subdifferentially continuous at $\bar{x}$. Then we may apply the next result at each point $(x, z) \in \operatorname{Graph} \partial f \cap B_\delta(\bar{x})$ to the function f_z and then take limits i.e.

$$D^* \partial f(\bar{x}, 0)(h) = \limsup_{\substack{(x,z) \to_{\operatorname{Graph} \partial_p f} (\bar{x}, 0) \\ h' \to h}} \hat{D}^* \partial_p f(x, z)(h').$$

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Theorem

Suppose $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is a prox-bounded, extended real valued, lower semi–continuous function. Suppose $\bar{x}$ is a strict local minimum order 2. Then we have $0 \in \partial_p f(\bar{x})$ and the following holds:
There exists some $\beta > 0$ such that for all $\|h\| = 1$ and all $p \in \hat{D}^* (\partial_p f)(x, 0)(h)$ we have $\langle h, p \rangle \geqslant \beta > 0$.

Moreover we have

$$(f_z)''_-(x, 0, h) \geqslant 0 \text{ for all } \|h\| = 1,$$

is certainly necessary for a local minimum of f_z at x.

- When we know f is convex then we have a number of interesting inversions formula.
- The conjugate of f is given by

$$f^*(z) := \inf_x \{\langle x, z \rangle - f(x)\}$$

and $z \in \partial f(x)$ iff $z \in (\partial f^*)^{-1}(x)$.

- Moreover for the coderivative we have

$$\begin{aligned} q \in D^*(\partial f)(x \mid z)(w) \qquad & \text{iff} \quad w \in D^*(\partial f^{-1})(z \mid x)(q) \\ & \text{iff} \quad w \in D^*(\partial f^*)(z \mid x)(q). \end{aligned}$$

- In particular

$$p \in D^* (\partial f) (\bar{x}, 0)(h) \text{ iff } h \in D^* (\partial f^*) (0, \bar{x})(p). \tag{8}$$

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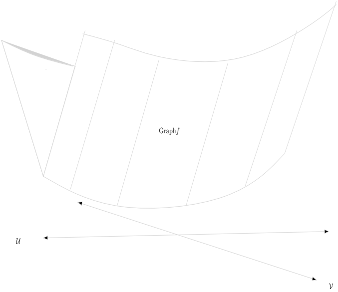
The $\mathcal{VU}$ decomposition

- When rel-int co $\partial f(\bar{x}) \neq \emptyset$ we can take

$$\bar{z} \in \text{rel-int co}\, \partial f(\bar{x}) \cap \partial f(\bar{x}).$$

- Define $\mathcal{V} := \text{span}\{\text{co}\, \partial f(\bar{x}) - \bar{z}\}$ and $\mathcal{U} := \mathcal{V}^{\perp}$.
- The idea here is that the subspace $\mathcal{V}$ contains the nonsmoothness while the subspace $\mathcal{U}$ contains the smooth part.



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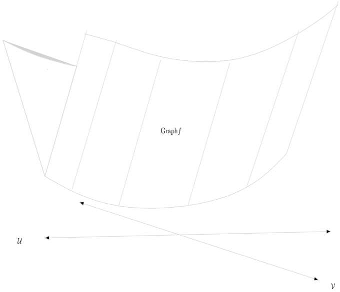

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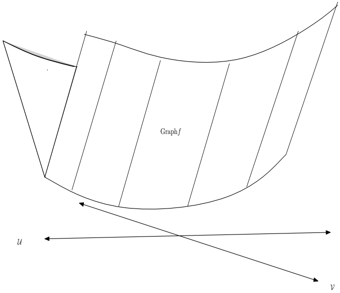

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- Under the $\mathcal{VU}$ decomposition we can then find $\varepsilon > 0$ such that

$$\bar{z} + B_{\varepsilon}(0) \cap \mathcal{V} \subseteq \operatorname{co} \partial f(\bar{x}) . \tag{9}$$

- One can then decompose $\bar{z} = \bar{z}_{\mathcal{U}} + \bar{z}_{\mathcal{V}}$ so that when $w = u + v \in \mathcal{U} \oplus \mathcal{V}$ we have $\langle \bar{z}, w \rangle = \langle \bar{z}_{\mathcal{U}}, u \rangle + \langle \bar{z}_{\mathcal{V}}, v \rangle$.
- Indeed we may decompose into the direct sum $\mathcal{U} \oplus \mathcal{V}$ and point $x = x_{\mathcal{U}} + x_{\mathcal{V}}$ and use the box norm for this decomposition $\|x - \bar{x}\| := \max \{\|x_{\mathcal{U}} - \bar{x}_{\mathcal{U}}\|, \|x_{\mathcal{V}} - \bar{x}_{\mathcal{V}}\|\}$.

Lemma

Consider $h : \mathcal{U} \to \mathbb{R}_{+\infty}$ is a proper lower semi-continuous function. Then $\partial_{\operatorname{co}} h(u) \subseteq \partial [\operatorname{co} h](u)$. When $\partial_{\operatorname{co}} h(u) \neq \emptyset$ then $\operatorname{co} h(u) = h(u)$ and we have $\partial_{\operatorname{co}} h(u) = \partial [\operatorname{co} h](u)$.

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- One can then decompose $\bar{z} = \bar{z}_{\mathcal{U}} + \bar{z}_{\mathcal{V}}$ so that when $w = u + v \in \mathcal{U} \oplus \mathcal{V}$ we have $\langle \bar{z}, w \rangle = \langle \bar{z}_{\mathcal{U}}, u \rangle + \langle \bar{z}_{\mathcal{V}}, v \rangle$.
- Indeed we may decompose into the direct sum $\mathcal{U} \oplus \mathcal{V}$ and point $x = x_{\mathcal{U}} + x_{\mathcal{V}}$ and use the box norm for this decomposition $\|x - \bar{x}\| := \max\{\|x_{\mathcal{U}} - \bar{x}_{\mathcal{U}}\|, \|x_{\mathcal{V}} - \bar{x}_{\mathcal{V}}\|\}$.

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Consider $h : \mathcal{U} \to \mathbb{R}_{+\infty}$ is a proper lower semi-continuous function. Then $\partial_{\operatorname{co}} h(u) \subseteq \partial[\operatorname{co} h](u)$. When $\partial_{\operatorname{co}} h(u) \neq \emptyset$ then $\operatorname{co} h(u) = h(u)$ and we have $\partial_{\operatorname{co}} h(u) = \partial[\operatorname{co} h](u)$.

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The $\mathcal{U}'$-Lagrangian

- Let $\mathcal{U}' \subseteq \mathcal{U}$ be a subspace, $\mathcal{V}' := (\mathcal{U}')^{\perp}$ and suppose $v(u) \in \arg\min_{v \in \mathcal{V}' \cap B_{\varepsilon}(0)} \{\partial f(\bar{x} + u + v) - \langle z_{\mathcal{V}'}, v \rangle\}$.
- Defined for $u \in \mathcal{U}'$ and $v(\cdot) : \mathcal{U}' \to \mathcal{V}' \cap B_{\varepsilon}(0)$ the axillary functions

$$\begin{aligned} k_v(u) &:= h(u + v(u)) - \langle \bar{z}_{\mathcal{V}'}, u + v(u) \rangle \\ \text{where} \quad h(w) &:= f(\bar{x} + w) + \delta_{[\mathcal{U}' \cap B_{\varepsilon}(0)] \oplus [\mathcal{V}' \cap B_{\varepsilon}(0)]}(w) \quad \text{and} \\ g(w) &:= \operatorname{co} h(w) \end{aligned}$$

- Then the $\mathcal{U}'$-Lagrangian

$$L^{\varepsilon}_{\mathcal{U}'}(u) := \inf_{v' \in \mathcal{V}'} \{h(u + v') - \langle \bar{z}_{\mathcal{V}}, v' \rangle\}.$$

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Proposition

Consider $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is a proper lower semi-continuous function and suppose that $\bar{x}$ give a tilt stable local minimum of f.

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The $\mathcal{U}'$-Lagrangian and Tilt Stability

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Some other second order quantities

Definition

1. The function f is said to be twice sub-differentiable (or possess a subjet) at x if the following set is nonempty;

$$\partial^{2,-} f(x) = \Big\{(z, Q) : f(x') \geqslant f(x) + \langle z, x' - x\rangle + \frac{1}{2}\langle Q(x' - x), (x' - x)\rangle + o(\|x' - x\|^2) \text{ for } x' \in B_\delta(x)\Big\}.$$

The subhessians at $(x, z) \in \operatorname{graph} \partial f$ are given by $\partial^{2,-} f(x, z) := \{Q \in \mathcal{S}(n) \mid (z, Q) \in \partial^{2,-} f(x)\}$.

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The $\mathcal{U}^2$ subspace of $\mathcal{U}$

We define the rank one barrier cone for $\underline{\partial}^2 f(x,z)$ as

$$b^1(\underline{\partial}^2 f(x,z)) := \{h \in \mathbb{R}^n \quad | \quad q\left(\underline{\partial}^2 f(x,z)\right)(h) := \sup\left\{\langle Qh, h\rangle \mid Q \in \underline{\partial}^2 f(x,z)\right\} < \infty\}.$$

and the second order component of $\mathcal{U}$ as $\mathcal{U}^2 := b^1(\underline{\partial}^2 f(x,z))$.

Lemma

Let the function $f : \mathbb{R}^n \mapsto \mathbb{R}_{+\infty}$ be finite at $\bar{x}$ and denote $\mathcal{U}^2 = b^1(\underline{\partial}^2 f(\bar{x},\bar{z}))$. Then

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Corollary

Suppose that f is quadratically minorized and is prox–regular at $\bar{x}$ for $\bar{z} \in \partial f(\bar{x})$ with respect to ε and r. Then $b^1(\underline{\partial}^2 f(\bar{x},\bar{z}))$ is a linear subspace of $\mathbb{R}^n$.

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- The function $v(\cdot)$ inherits the uniqueness of $m(\cdot)$.
- The function $k_v(\cdot)$ being well defined also inherits the tilt stability of the local minimum $\bar{x}$ of f but is shifted to $0 \in \mathcal{U}$.
- We have $\operatorname{co} k_v$ inherting tilt stability from k_v and so $q \neq 0$

$$\langle p, q \rangle > 0 \quad \text{for all} \quad p \in D^* \left(\partial \left[\operatorname{co} k_v \right] \right) (0|0) \left(q \right).$$

- Tilt stability can be shown to be equivalent to there being a stable strong local minimizers of $\operatorname{co} k_v$ at 0.
- We say $(\operatorname{co} k_v)_z := \operatorname{co} k_v - \langle z, \cdot \rangle$ has a strict local minimum order two at u' relative to $B_\delta(0)$ when $(\operatorname{co} k_v)_z(u) \geqslant (\operatorname{co} k_v)_z \left(u' \right) + \beta \left\| u - u' \right\|^2$ for all $u', u \in B_\delta(0)$.

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for all z, $z' \in B_\delta(0) \cap \mathcal{U}^2$.

- This later property is known to forces single valuedness of $(\partial \operatorname{co} k_v)^{-1} = \partial(\operatorname{co} k_v)^*$ locally in some small ball. Indeed $u \mapsto \nabla(\operatorname{co} k_v)^* = \nabla k_v^*$ exists and is Lipchitz at a rate L.

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- This later property is known to forces single valuedness of $(\partial \operatorname{co} k_v)^{-1} = \partial(\operatorname{co} k_v)^*$ locally in some small ball. Indeed $u \mapsto \nabla(\operatorname{co} k_v)^* = \nabla k_v^*$ exists and is Lipchitz at a rate L.

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- The bounded second order behaviour on $\mathcal{U}^2$ is critical here, in ruling out a singularity of the inverse coderivative at $z = 0$.
- This in turn implies the same conclusion for

$$u \mapsto \mathrm{co}\, f(\bar{x} + u + v(u)) \quad \text{for } u \in \mathcal{U}^2.$$

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The First Main Result

Theorem

Consider $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ *is a proper lower semi-continuous function, quadratically minorized and a prox-regular function at* $\bar{x}$ *for* $0 \in \partial f(\bar{x})$*. Suppose in addition* f *admits a nontrivial subspace* $\mathcal{U}^2 := b^1\left(\underline{\partial}^2 f(\bar{x}, 0)\right) \subseteq \mathcal{U}$*. Suppose that* f *has a tilt stable local minimum at* $\bar{x}$ *and let* $g(w) := [\operatorname{co} h](w)$ *and* $\{v(u)\} = \arg\min_{v' \in \mathcal{V}^2 \cap B_\varepsilon(0)} f(\bar{x} + u + v') : \mathcal{U}^2 \to \mathcal{V}^2$*.*
Then

- *we have* $g(u + v(u)) = f(\bar{x} + u + v(u))$ *and* $\nabla_u g(u + v(u))$ *existing as Lipschitz functions for* $u \in B_\delta^{\mathcal{U}^2}(0)$*,*
- *moreover* $\mathcal{M} := \left\{(u, v(u)) \mid u \in B_\varepsilon^{\mathcal{U}^2}(0)\right\}$ *is a manifold on which the restriction to* $\mathcal{M}$ *of* f *is smooth.*

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