Stationarity and Transversality of Finite and Infinite Collections of Sets

Alexander Kruger   Marco López Cerdá

Centre for Informatics and Applied Optimization,
Faculty of Science and Technology
Federation University Australia, Ballarat
a.kruger@federation.edu.au

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Collections of Sets

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  - Extremal principle (K., Mordukhovich, 1980)
  - Boundary condition, nonconvex separation property (Borwein, Jofré, 1998)
  - Jamesons property (G) (1972)
  - Metric inequality, subtransversality (Dolecki, 1982; Ioffe, 1989, 2015; Ngai, Théra, 2001)
  - (Strong) conical hull intersection property (Chui, Deutsch, Ward, 1992; Deutsch, Li, Ward; 1997)

Outline

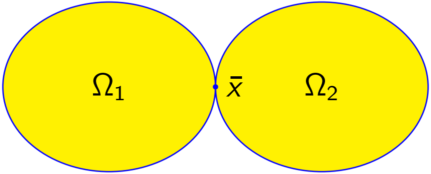
1. Extremality and Extremal Principle

2. Stationarity vs Transversality

3. Infinite Collections

Outline

Extremality



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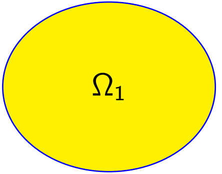

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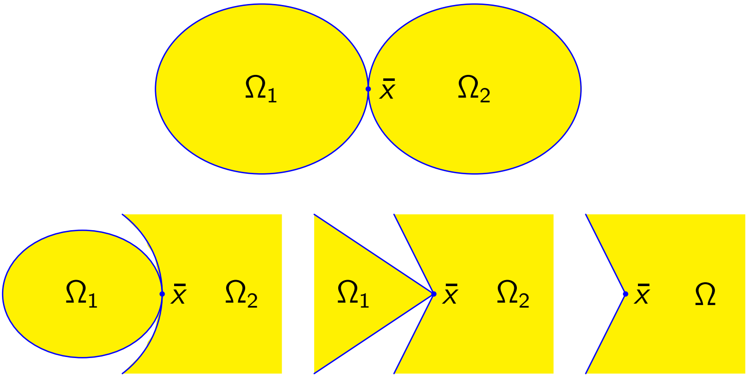

Extremality

X – Banach space,
$\boldsymbol{\Omega} := \{\Omega_i\}_{i\in I} \subset X, \quad 1 < |I| < \infty, \quad \bar{x} \in \bigcap_{i\in I} \Omega_i$

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$\mathbf{\Omega}$ is *locally extremal* at $\bar{x}$ if $\exists\delta > 0\ \forall\varepsilon > 0\ \exists a_i \in \varepsilon\mathbb{B}\ (i \in I)$ s.t.

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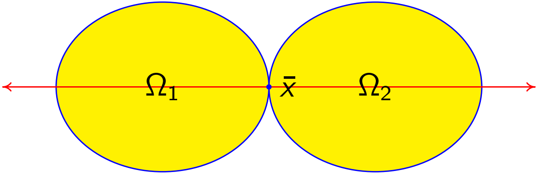
Extremality

Dual Characterization



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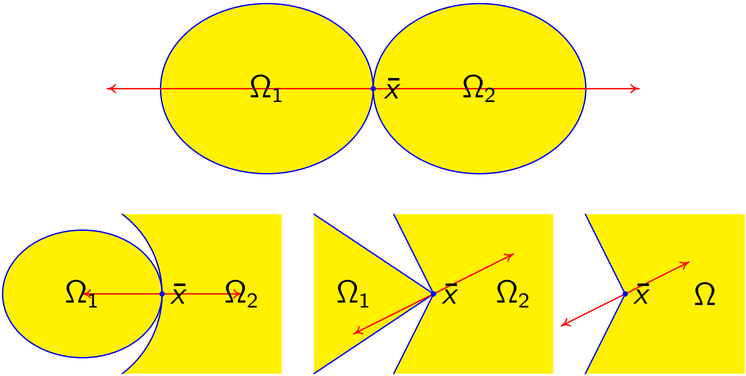

Fréchet Normal Cone

$\bar{x} \in \Omega$
Fréchet normal cone:

$$N_\Omega(\bar{x}) = \left\{ x^* \in X^* : \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \le 0 \right\}$$

Extremal Principle

Ω_i $(i \in I)$ are closed

Extremal Principle (K., Mordukhovich, 1980)

If $\mathbf{\Omega}$ is locally extremal at $\bar{x}$ then $\forall \varepsilon > 0$ $\exists x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $x_i^* \in N_{\Omega_i}(x_i)$ $(i \in I)$ s.t.

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Theorem (Mordukhovich, Shao, 1996)

Extremal Principle holds if and only if X is Asplund

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Approximate Stationarity

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Local extremality $\quad\Rightarrow\quad$ approximate stationarity

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Theorem (K., 2003)

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Transversality

Definition (K., 1998, 2009)

$\mathbf{\Omega}$ is *approximately stationary* at $\bar{x}$ if $\forall \varepsilon > 0\ \exists \rho \in (0, \varepsilon)$, $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$, $a_i \in \alpha\rho\mathbb{B}$ $(i \in I)$ s.t.

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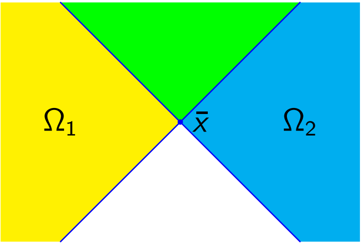
Moreover, $\mathrm{tr}[\mathbf{\Omega}](\bar{x}) = \sup\{\alpha$ in the above theorems$\}$

Transversality vs Stationarity



$$\mathrm{tr}[\boldsymbol{\Omega}](\bar{x}) = \frac{\sqrt{2}}{2}$$

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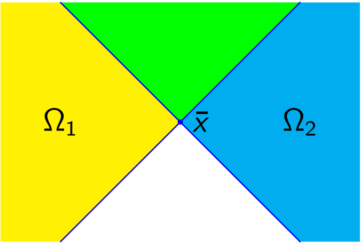

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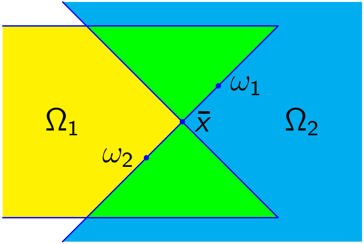

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X – Asplund space, Ω_i $(i \in I)$ – closed, $\bar{x} \in \bigcap_{i \in I} \Omega_i$

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Moreover, $\mathrm{tr}[\mathbf{\Omega}](\bar{x}) = \sup\{\alpha$ *in the above inequality*$\}$

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Thank you